Mathematics Learning Centre

The University of Sydney

Functions and Their Graphs

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1 Functions

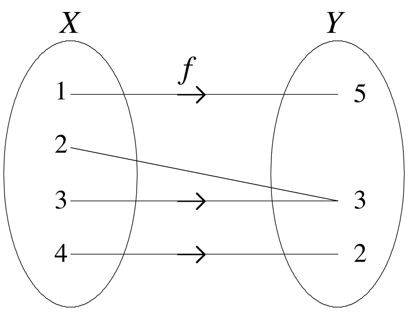
In this Chapter we will cover various aspects of functions. We will look at the definition of a function, the domain and range of a function, what we mean by specifying the domain of a function and absolute value function.

1.1 What is a function?

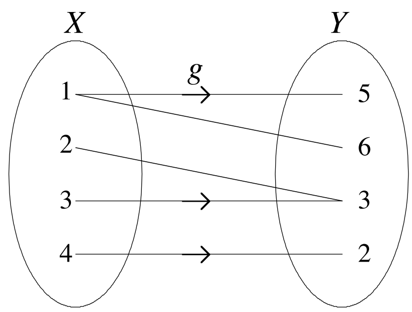
1.1.1 Definition of a function

> A function f from a set of elements X to a set of elements Y is a rule that assigns to each element x in X exactly one element y in Y.

One way to demonstrate the meaning of this definition is by using arrow diagrams.



$f : X \to Y$ is a function. Every element in X has associated with it exactly one element of Y.



$g : X \to Y$ is not a function. The element 1 in set X is assigned two elements, 5 and 6 in set Y.

A function can also be described as a set of ordered pairs (x, y) such that for any x-value in the set, there is only one y-value. This means that there cannot be any repeated x-values with different y-values.

The examples above can be described by the following sets of ordered pairs.

F = {(1,5),(3,3),(2,3),(4,2)} is a function.

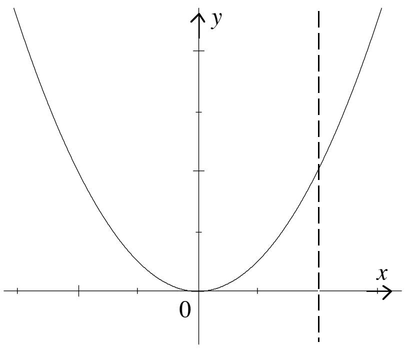
G = {(1,5),(4,2),(2,3),(3,3),(1,6)} is not a function.

The definition we have given is a general one. While in the examples we have used numbers as elements of X and Y, there is no reason why this must be so. However, in these notes we will only consider functions where X and Y are subsets of the real numbers.

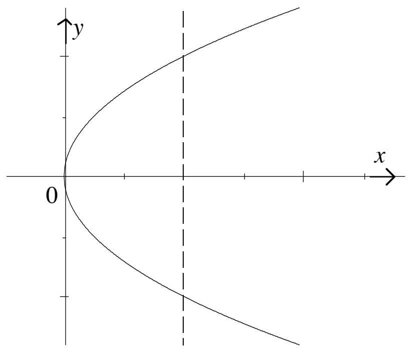
In this setting, we often describe a function using the rule, $y = f(x)$, and create a graph of that function by plotting the ordered pairs $(x, f(x))$ on the Cartesian Plane. This graphical representation allows us to use a test to decide whether or not we have the graph of a function: The Vertical Line Test.

1.1.2 The Vertical Line Test

The Vertical Line Test states that if it is *not possible* to draw a vertical line through a graph so that it cuts the graph in more than one point, then the graph *is* a function.



This is the graph of a function. All possible vertical lines will cut this graph only once.



This is not the graph of a function. The vertical line we have drawn cuts the graph twice.

1.1.3 Domain of a function

For a function $f : X \to Y$ the *domain* of f is the set X.

This also corresponds to the set of x-values when we describe a function as a set of ordered pairs (x, y).

If only the rule $y = f(x)$ is given, then the domain is taken to be the set of all real x for which the function is defined. For example, $y = \sqrt{x}$ has domain; all real $x \geq 0$. This is sometimes referred to as the *natural* domain of the function.

1.1.4 Range of a function

For a function $f : X \to Y$ the *range* of f is the set of y-values such that $y = f(x)$ for some x in X.

This corresponds to the set of y-values when we describe a function as a set of ordered pairs (x, y). The function $y = \sqrt{x}$ has range; all real $y \geq 0$.

Example

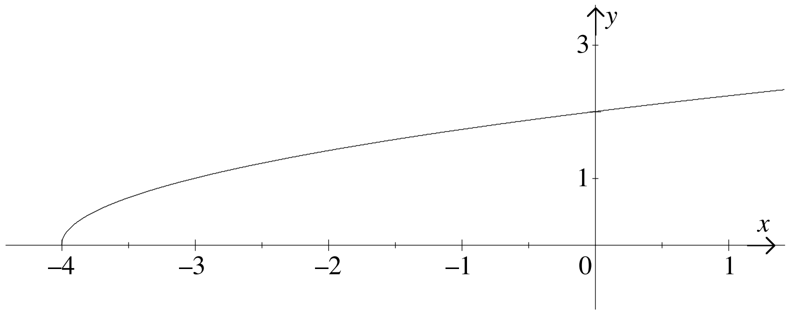
a. State the domain and range of $y = \sqrt{x + 4}$.

b. Sketch, showing significant features, the graph of $y = \sqrt{x + 4}$.

Solution

a. The domain of $y = \sqrt{x+4}$ is all real $x \geq -4$. We know that square root functions are only defined for positive numbers so we require that $x + 4 \geq 0$, ie $x \geq -4$. We also know that the square root functions are always positive so the range of $y = \sqrt{x+4}$ is all real $y \geq 0$.

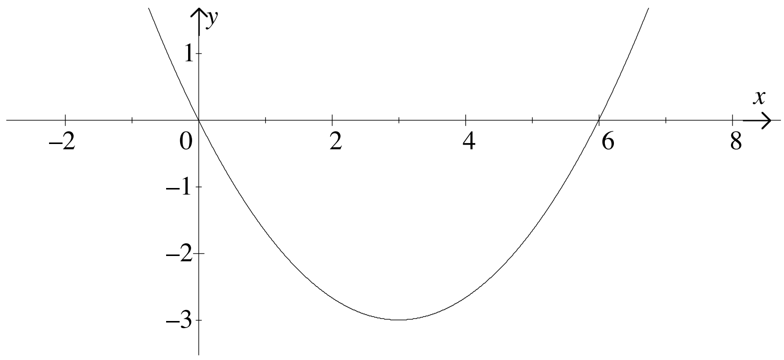
b.



The graph of $y = \sqrt{x+4}$.

Example

a. State the equation of the parabola sketched below, which has vertex $(3, -3)$.



b. Find the domain and range of this function.

Solution

a. The equation of the parabola is $y = \frac{x^2-6x}{3}$.

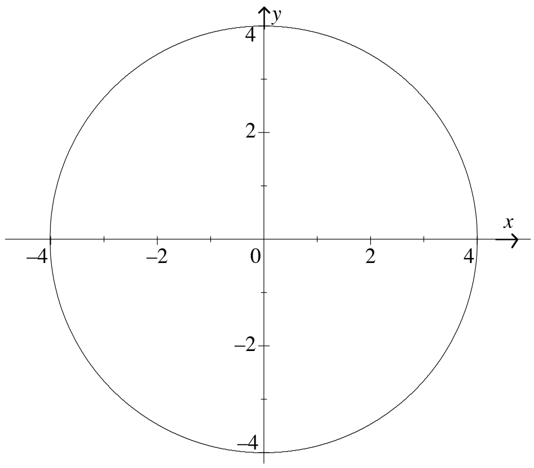
b. The domain of this parabola is all real x. The range is all real $y \geq -3$.

Example

Sketch $x^2 + y^2 = 16$ and explain why it is not the graph of a function.

Solution

$x^2 + y^2 = 16$ is not a function as it fails the vertical line test. For example, when $x = 0$ $y = \pm 4$.



The graph of $x^2 + y^2 = 16$.

Example

Sketch the graph of $f(x) = 3x - x^2$ and find

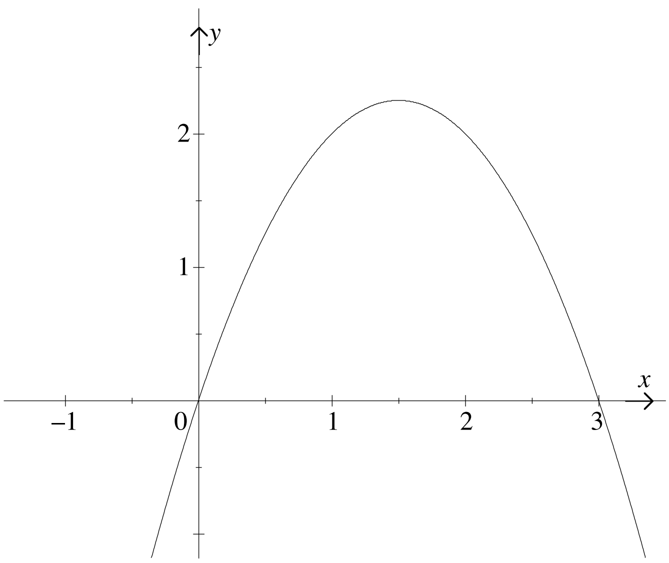
a. the domain and range

b. $f(q)$

c. $f(x^2)$

d. $\frac{f(2+h)-f(2)}{h}$, $h \neq 0$.

Solution



The graph of $f(x) = 3x - x^2$.

a. The domain is all real x. The range is all real y where $y \leq 2.25$.

b. $f(q) = 3q - q^2$

c. $f(x^2) = 3(x^2) - (x^2)^2 = 3x^2 - x^4$

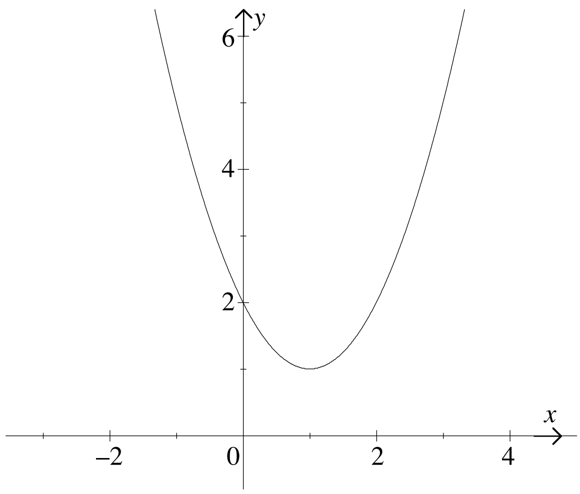
d.

$$\begin{aligned}\frac{f(2+h)-f(2)}{h} &= \frac{(3(2+h)-(2+h)^2)-(3(2)-(2)^2)}{h}\\ &= \frac{6+3h-(h^2+4h+4)-2}{h}\\ &= \frac{-h^2-h}{h}\\ &= -h-1\end{aligned}$$

Example

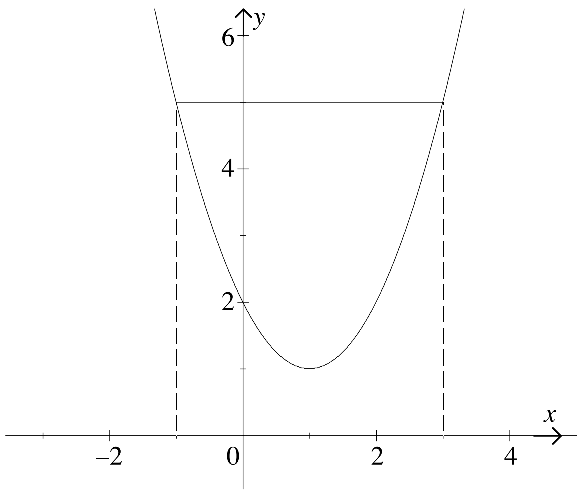
Sketch the graph of the function $f(x) = (x-1)^2 + 1$ and show that $f(p) = f(2-p)$. Illustrate this result on your graph by choosing one value of p.

Solution



The graph of $f(x) = (x-1)^2 + 1$.

$$\begin{aligned}f(2-p) &= ((2-p)-1)^2+1\\ &= (1-p)^2+1\\ &= (p-1)^2+1\\ &= f(p)\end{aligned}$$



The sketch illustrates the relationship $f(p) = f(2-p)$ for $p = -1$. If $p = -1$ then $2 - p = 2 - (-1) = 3$, and $f(-1) = f(3)$.

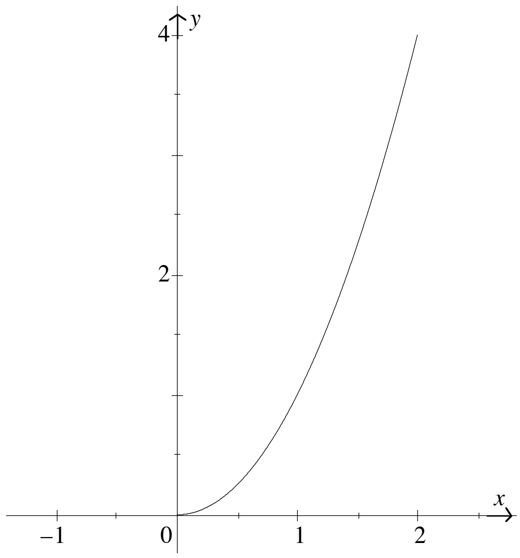
1.2 Specifying or restricting the domain of a function

We sometimes give the rule $y = f(x)$ along with the domain of definition. This domain may not necessarily be the natural domain. For example, if we have the function

$$y = x^2 \quad \text{for} \quad 0 \leq x \leq 2$$

then the domain is given as $0 \leq x \leq 2$. The natural domain has been restricted to the subinterval $0 \leq x \leq 2$.

Consequently, the range of this function is all real y where $0 \leq y \leq 4$. We can best illustrate this by sketching the graph.



The graph of $y = x^2$ for $0 \leq x \leq 2$.

1.3 The absolute value function

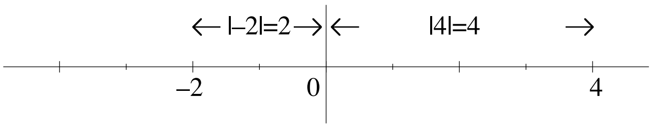
Before we define the absolute value function we will review the definition of the absolute value of a number.

The *Absolute value of a number* x is written $|x|$ and is defined as

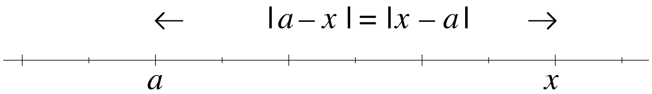
$$|x| = x \text{ if } x \geq 0 \qquad \text{or} \qquad |x| = -x \text{ if } x < 0.$$

That is, $|4| = 4$ since 4 is positive, but $|-2| = 2$ since -2 is negative.

We can also think of $|x|$ geometrically as the distance of x from 0 on the number line.



More generally, $|x - a|$ can be thought of as the distance of x from a on the numberline.



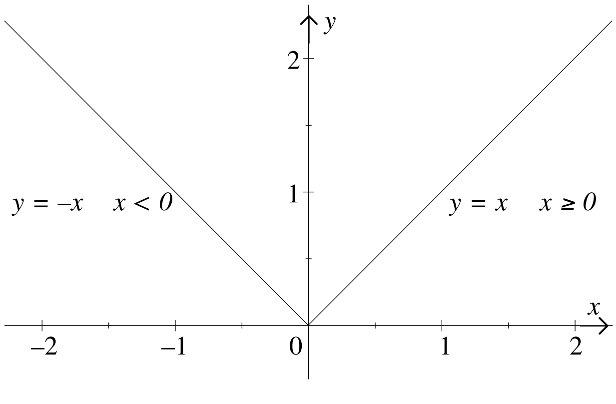
Note that $|a - x| = |x - a|$.

The absolute value *function* is written as $y = |x|$.

We define this function as

$$y = \begin{cases} +x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

From this definition we can graph the function by taking each part separately. The graph of $y = |x|$ is given below.



The graph of $y = |x|$.

Example

Sketch the graph of $y = |x - 2|$.

Solution

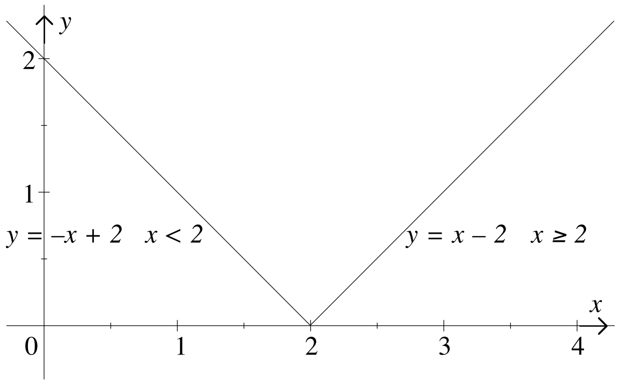
For $y = |x - 2|$ we have

$$y = \begin{cases} +(x-2) & \text{when } x - 2 \geq 0 \quad \text{or} \quad x \geq 2 \\ -(x-2) & \text{when } x - 2 < 0 \quad \text{or} \quad x < 2 \end{cases}$$

That is,

$$y = \begin{cases} x - 2 & \text{for } x \geq 2 \\ -x + 2 & \text{for } x < 2 \end{cases}$$

Hence we can draw the graph in two parts.



The graph of $y = |x - 2|$.

We could have sketched this graph by first of all sketching the graph of $y = x - 2$ and then reflecting the negative part in the x-axis. We will use this fact to sketch graphs of this type in Chapter 2.

1.4 Exercises

1. **a.** State the domain and range of $f(x) = \sqrt{9 - x^2}$.

b. Sketch the graph of $y = \sqrt{9 - x^2}$.

2. Given $\psi(x) = x^2 + 5$, find, in simplest form, $\dfrac{\psi(x+h) - \psi(x)}{h}$ $\quad h \neq 0$.

3. Sketch the following functions stating the domain and range of each:

a. $y = \sqrt{x - 1}$

b. $y = |2x|$

c. $y = \frac{1}{x-4}$

d. $y = |2x| - 1$.

4. **a.** Find the perpendicular distance from $(0, 0)$ to the line $x + y + k = 0$

b. If the line $x + y + k = 0$ cuts the circle $x^2 + y^2 = 4$ in two distinct points, find the restrictions on k.

5. Sketch the following, showing their important features.

a. $y = \left(\frac{1}{2}\right)^x$

b. $y^2 = x^2$.

6. Explain the meanings of function, domain and range. Discuss whether or not $y^2 = x^3$ is a function.

7. Sketch the following relations, showing all intercepts and features. State which ones are functions giving their domain and range.

a. $y = -\sqrt{4 - x^2}$

b. $|x| - |y| = 0$

c. $y = x^3$

d. $y = \frac{x}{|x|}, x \neq 0$

e. $|y| = x$.

8. If $A(x) = x^2 + 2 + \frac{1}{x^2}$, $x \neq 0$, prove that $A(p) = A(\frac{1}{p})$ for all $p \neq 0$.

9. Write down the values of x which are not in the domain of the following functions:

a. $f(x) = \sqrt{x^2 - 4x}$

b. $g(x) = \frac{x}{x^2-1}$

10. If $\phi(x) = \log\left(\frac{x}{x-1}\right)$, find in simplest form:

a. $\phi(3) + \phi(4) + \phi(5)$

b. $\phi(3) + \phi(4) + \phi(5) + \cdots + \phi(n)$

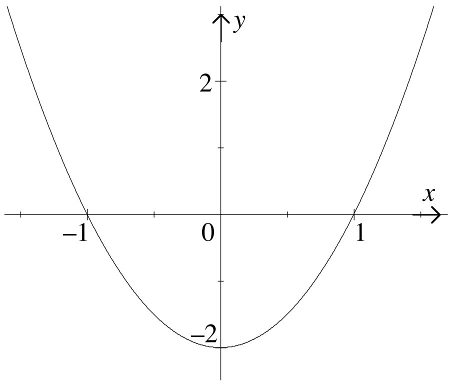
11. **a.** If $y = x^2 + 2x$ and $x = (z - 2)^2$, find y when $z = 3$.

b. Given $L(x) = 2x + 1$ and $M(x) = x^2 - x$, find

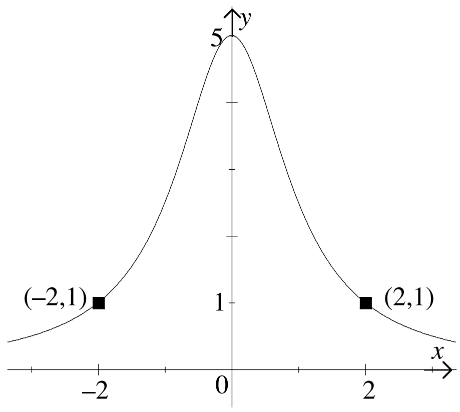
i $L(M(x))$

ii $M(L(x))$

12. Using the sketches, find the value(s) of the constants in the given equations:



$y = ax^2 - b$



$y = \frac{a}{bx^2+1}$

13. a. Define $|a|$, the absolute value of a, where a is real.

b. Sketch the relation $|x| + |y| = 1$.

14. Given that $S(n) = \frac{n}{2n+1}$, find an expression for $S(n-1)$.

Hence show that $S(n) - S(n-1) = \frac{1}{(2n-1)(2n+1)}$.

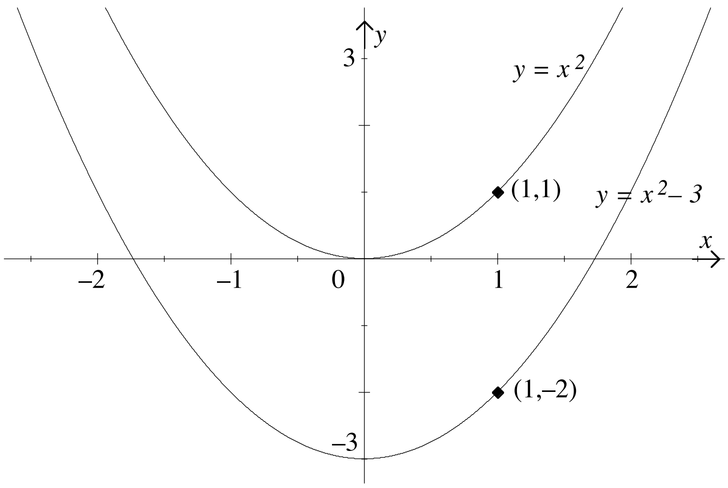
2 More about functions

In this Chapter we will look at the effects of stretching, shifting and reflecting the basic functions, $y = x^2$, $y = x^3$, $y = \frac{1}{x}$, $y = |x|$, $y = a^x$, $x^2 + y^2 = r^2$. We will introduce the concepts of even and odd functions, increasing and decreasing functions and will solve equations using graphs.

2.1 Modifying functions by shifting

2.1.1 Vertical shift

We can draw the graph of $y = f(x) + k$ from the graph of $y = f(x)$ as the addition of the constant k produces a *vertical shift*. That is, adding a constant to a function moves the graph up k units if $k > 0$ or down k units if $k < 0$. For example, we can sketch the function $y = x^2 - 3$ from our knowledge of $y = x^2$ by shifting the graph of $y = x^2$ *down* by 3 units. That is, if $f(x) = x^2$ then $f(x) - 3 = x^2 - 3$.

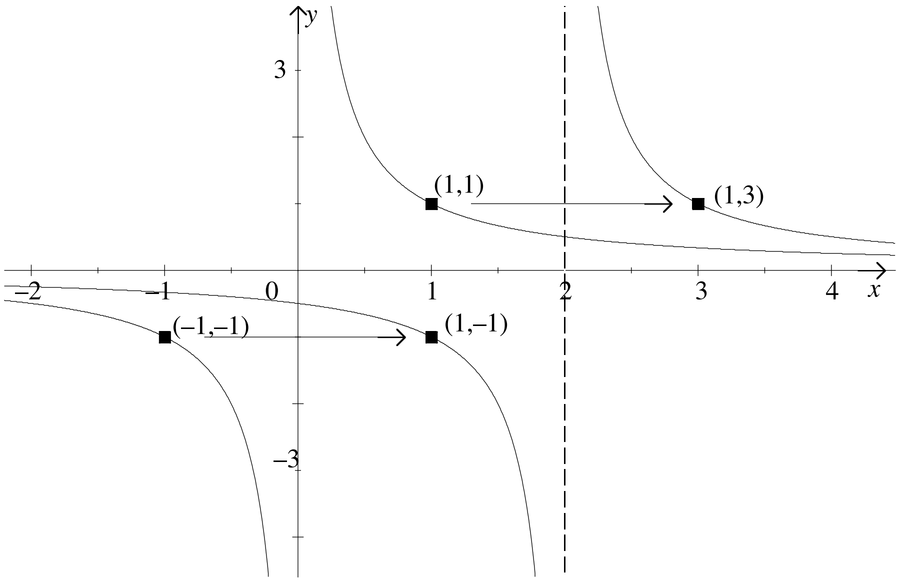


We can also write $y = f(x) - 3$ as $y + 3 = f(x)$, so replacing y by $y + 3$ in $y = f(x)$ also shifts the graph down by 3 units.

2.1.2 Horizontal shift

We can draw the graph of $y = f(x - a)$ if we know the graph of $y = f(x)$ as placing the constant a inside the brackets produces a *horizontal shift*. If we replace x by $x - a$ inside the function then the graph will shift to the left by a units if $a < 0$ and to the right by a units if $a > 0$.

For example we can sketch the graph of $y = \frac{1}{x-2}$ from our knowledge of $y = \frac{1}{x}$ by shifting this graph to the right by 2 units. That is, if $f(x) = \frac{1}{x}$ then $f(x-2) = \frac{1}{x-2}$.

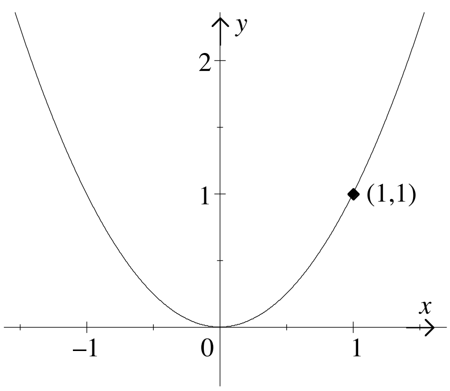


Note that the function $y = \frac{1}{x-2}$ is not defined at $x = 2$. The point $(1,1)$ has been shifted to $(1,3)$.

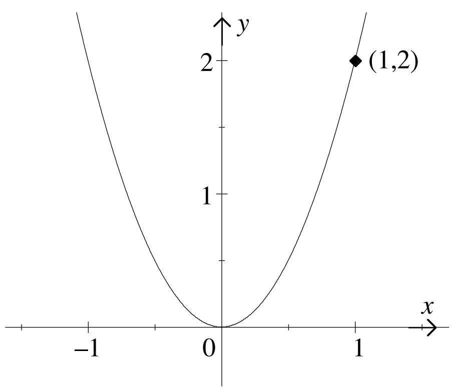
2.2 Modifying functions by stretching

We can sketch the graph of a function $y = bf(x)$ $(b > 0)$ if we know the graph of $y = f(x)$ as multiplying by the constant b will have the effect of stretching the graph in the y-direction by a factor of b. That is, multiplying $f(x)$ by b will change all of the y-values proportionally.

For example, we can sketch $y = 2x^2$ from our knowledge of $y = x^2$ as follows:

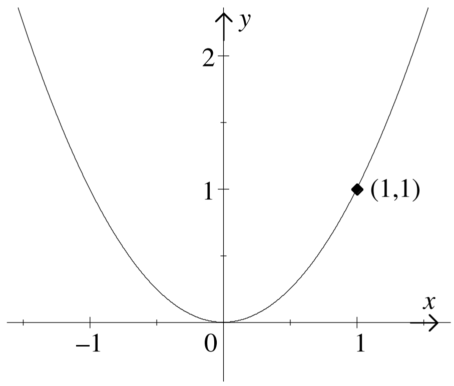


The graph of $y = x^2$.

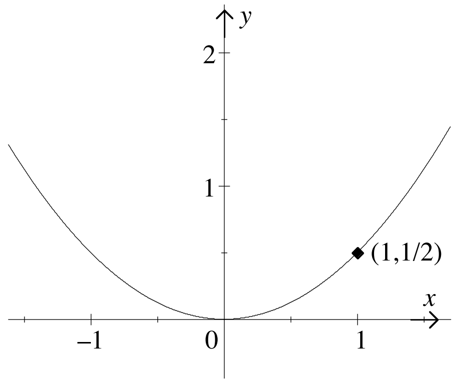


The graph of $y = 2x^2$. Note, all the y-values have been multiplied by 2, but the x-values are unchanged.

We can sketch the graph of $y = \frac{1}{2}x^2$ from our knowledge of $y = x^2$ as follows:



The graph of $y = x^2$.

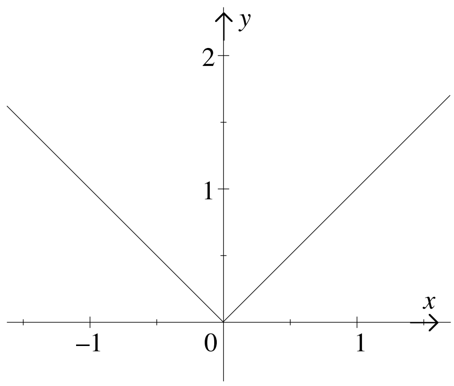


The graph of $y = \frac{1}{2}x^2$. Note, all the y-values have been multiplied by $\frac{1}{2}$, but the x-values are unchanged.

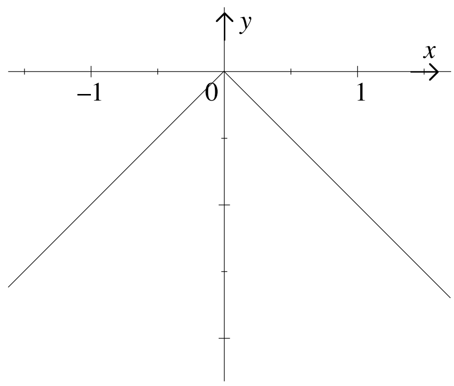
2.3 Modifying functions by reflections

2.3.1 Reflection in the x-axis

We can sketch the function $y = -f(x)$ if we know the graph of $y = f(x)$, as a minus sign in front of $f(x)$ has the effect of reflecting the whole graph in the x-axis. (Think of the x-axis as a mirror.) For example, we can sketch $y = -|x|$ from our knowledge of $y = |x|$.



The graph of $y = |x|$.

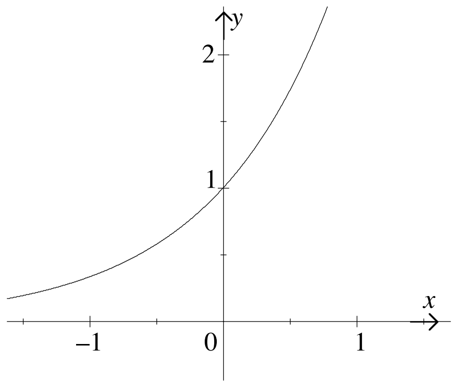


The graph of $y = -|x|$. It is the reflection of $y = |x|$ in the x-axis.

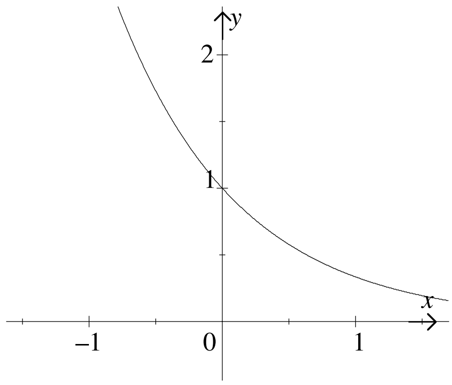
2.3.2 Reflection in the y-axis

We can sketch the graph of $y = f(-x)$ if we know the graph of $y = f(x)$ as the graph of $y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis.

For example, we can sketch $y = 3^{-x}$ from our knowledge of $y = 3^x$.



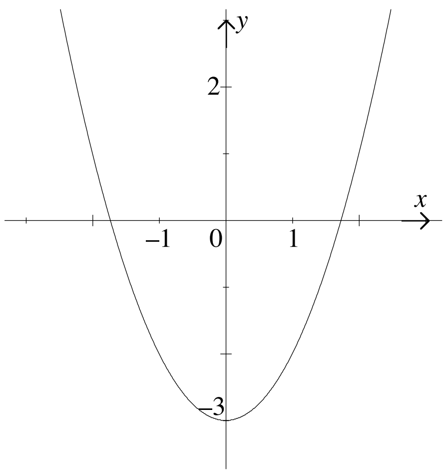
The graph of $y = 3^x$.



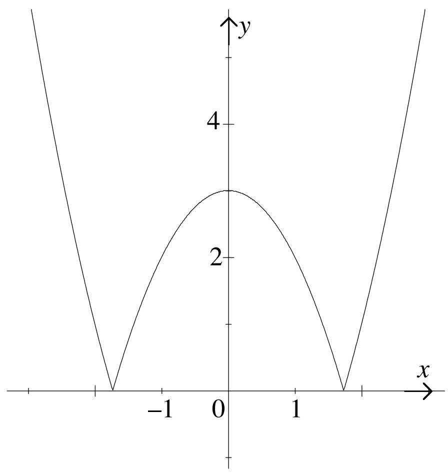
The graph of $y = 3^{-x}$. It is the reflection of $y = 3^x$ in the y-axis.

2.4 Other effects

We can sketch the graph of $y = |f(x)|$ if we know the graph of $y = f(x)$ as the effect of the absolute value is to reflect all of the *negative* values of $f(x)$ in the x-axis. For example, we can sketch the graph of $y = |x^2 - 3|$ from our knowledge of the graph of $y = x^2 - 3$.



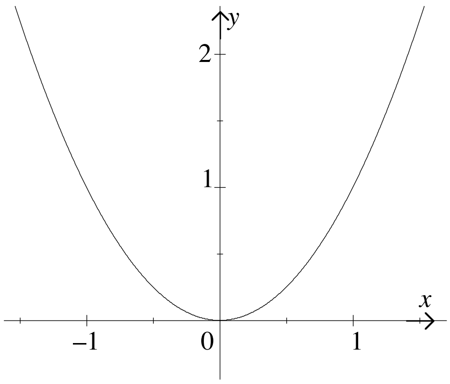
The graph of $y = x^2 - 3$.



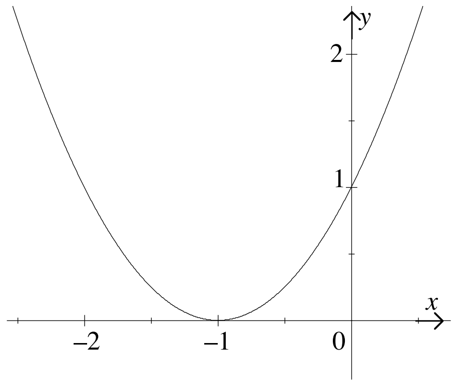
The graph of $y = |x^2 - 3|$. The negative values of $y = x^2 - 3$ have been reflected in the x-axis.

2.5 Combining effects

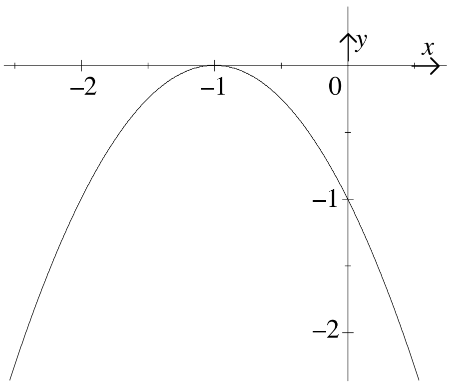
We can use all the above techniques to graph more complex functions. For example, we can sketch the graph of $y = 2 - (x + 1)^2$ from the graph of $y = x^2$ provided we can analyse the combined effects of the modifications. Replacing x by $x + 1$ (or $x - (-1)$) moves the graph to the left by 1 unit. The effect of the $-$ sign in front of the brackets turns the graph up side down. The effect of adding 2 moves the graph up 2 units. We can illustrate these effects in the following diagrams.



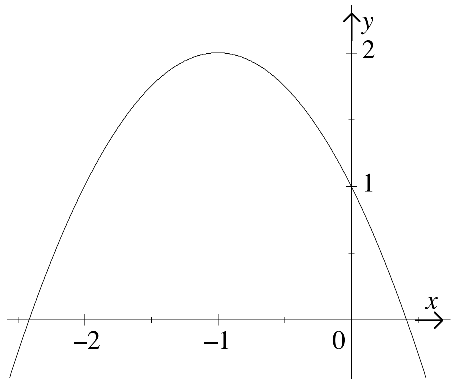
The graph of $y = x^2$.



The graph of $y = (x+1)^2$. The graph of $y = x^2$ has been shifted 1 unit to the left.



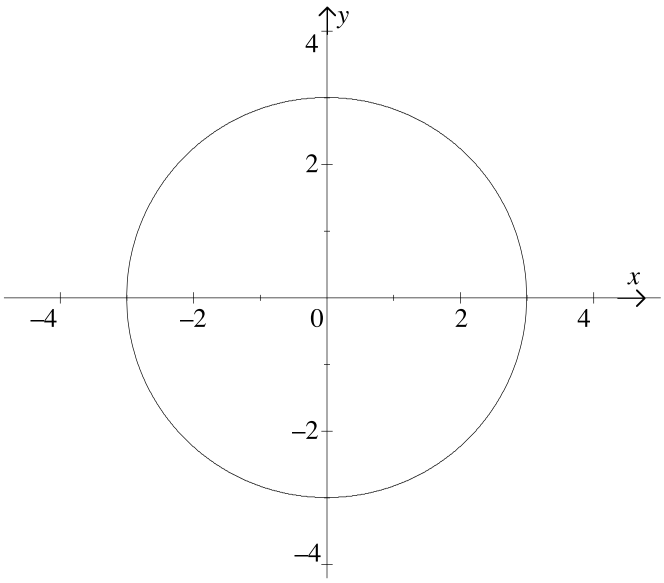
The graph of $y = -(x+1)^2$. The graph of $y = (x+1)^2$ has been reflected in the x-axis.



The graph of $y = 2-(x+1)^2$. The graph of $y = -(x+1)^2$ has been shifted up by 2 units.

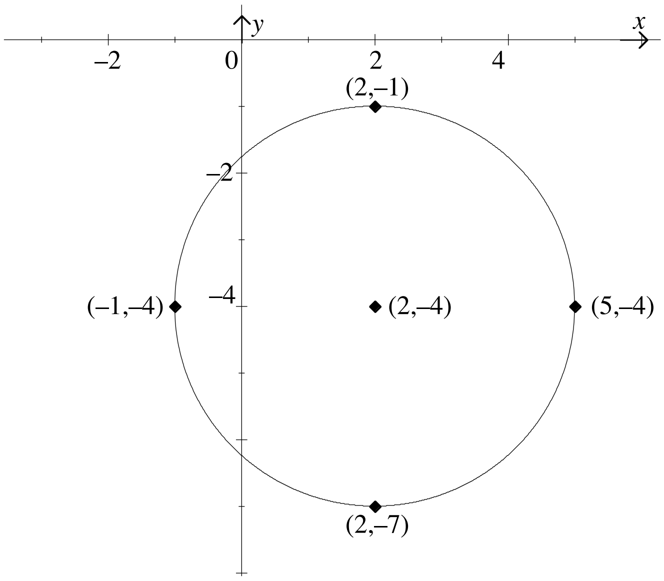
Similarly, we can sketch the graph of $(x-h)^2+(y-k)^2 = r^2$ from the graph of $x^2+y^2 = r^2$. Replacing x by $x-h$ shifts the graph sideways h units. Replacing y by $y-k$ shifts the graph up or down k units. (We remarked before that $y = f(x)+k$ could be written as $y-k = f(x)$.)

For example, we can use the graph of the circle of radius 3, $x^2+y^2 = 9$, to sketch the graph of $(x-2)^2+(y+4)^2 = 9$.



The graph of $x^2 + y^2 = 9$.

This is a circle centre $(0, 0)$, radius 3.



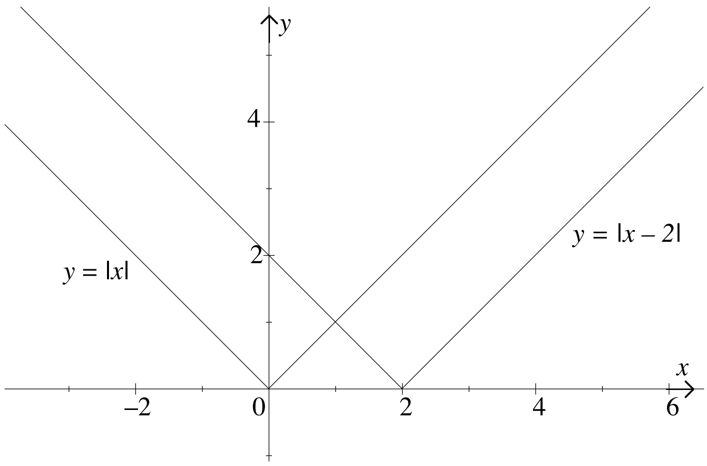
The graph of $(x - 2)^2 + (y + 4)^2 = 9$.

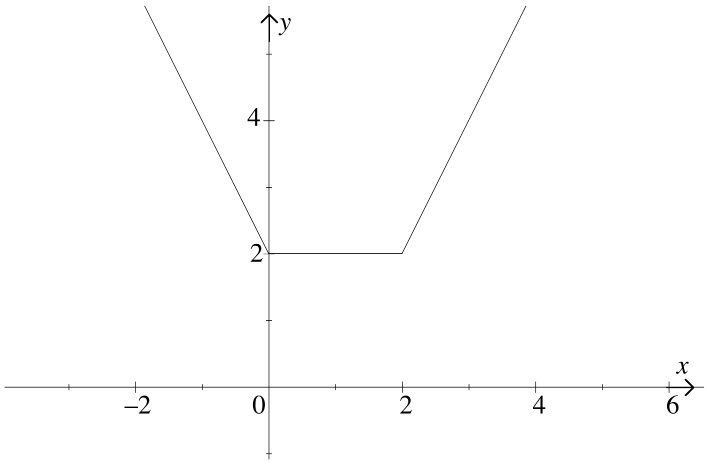
This is a circle centre $(2, -4)$, radius 3.

Replacing x by $x - 2$ has the effect of shifting the graph of $x^2 + y^2 = 9$ two units to the right. Replacing y by $y + 4$ shifts it down 4 units.

2.6 Graphing by addition of ordinates

We can sketch the graph of functions such as $y = |x| + |x - 2|$ by drawing the graphs of both $y = |x|$ and $y = |x - 2|$ on the same axes then adding the corresponding y-values.





The graph of $y = |x| + |x - 2|$.

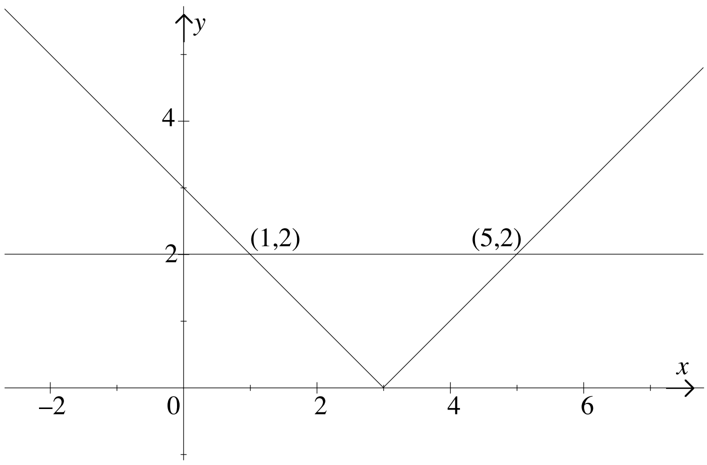
At each point of x the y-values of $y = |x|$ and $y = |x - 2|$ have been added. This allows us to sketch the graph of $y = |x| + |x - 2|$.

This technique for sketching graphs is very useful for sketching the graph of the sum of two trigonometric functions.

2.7 Using graphs to solve equations

We can solve equations of the form $f(x) = k$ by sketching $y = f(x)$ and the horizontal line $y = k$ on the same axes. The solution to the equation $f(x) = k$ is found by determining the x-values of any points of intersection of the two graphs.

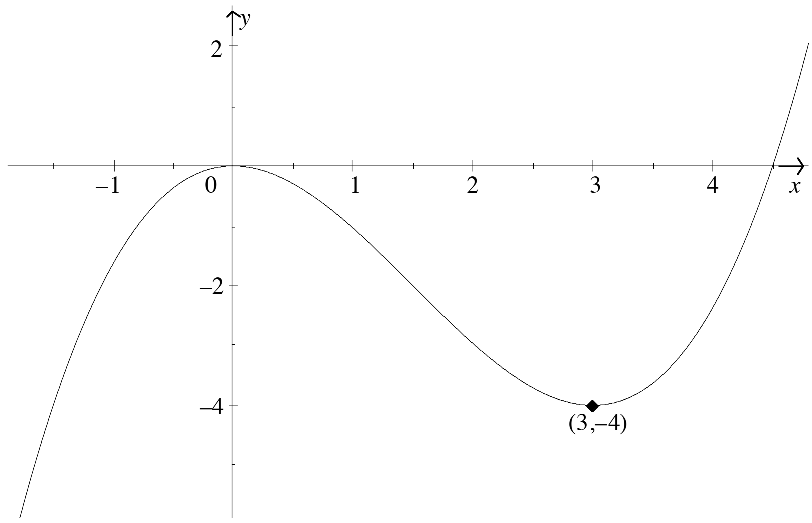
For example, to solve $|x-3|=2$ we sketch $y=|x-3|$ and $y=2$ on the same axes.



The x-values of the points of intersection are 1 and 5. Therefore $|x-3|=2$ when $x=1$ or $x=5$.

Example

The graph of $y=f(x)$ is sketched below.



For what values of k does the equation $f(x)=k$ have

1. 1 solution

2. 2 solutions

3. 3 solutions?

Solution

If we draw a horizontal line $y=k$ across the graph $y=f(x)$, it will intersect once when $k>0$ or $k<-4$, twice when $k=0$ or $k=-4$ and three times when $-4<k<0$. Therefore the equation $f(x)=k$ will have

1. 1 solution if $k > 0$ or $k < -4$

2. 2 solutions if $k = 0$ or $k = -4$

3. 3 solutions if $-4 < k < 0$.

2.8 Exercises

1. Sketch the following:

a. $y = x^2$ **b.** $y = \frac{1}{3}x^2$ **c.** $y = -x^2$ **d.** $y = (x+1)^2$

2. Sketch the following:

a. $y = \frac{1}{x}$ **b.** $y = \frac{1}{x-2}$ **c.** $y = \frac{-2}{x}$ **d.** $y = \frac{1}{x+1} + 2$

3. Sketch the following:

a. $y = x^3$ **b.** $y = |x^3 - 2|$ **c.** $y = 3 - (x-1)^3$

4. Sketch the following:

a. $y = |x|$ **b.** $y = 2|x-2|$ **c.** $y = 4 - |x|$

5. Sketch the following:

a. $x^2 + y^2 = 16$ **b.** $x^2 + (y+2)^2 = 16$ **c.** $(x-1)^2 + (y-3)^2 = 16$

6. Sketch the following:

a. $y = \sqrt{9 - x^2}$ **b.** $y = \sqrt{9 - (x-1)^2}$ **c.** $y = \sqrt{9 - x^2} - 3$

7. Show that $\dfrac{x-1}{x-2} = \dfrac{1}{x-2} + 1$.

Hence sketch the graph of $y = \dfrac{x-1}{x-2}$.

8. Sketch $y = \frac{x+1}{x-1}$.

9. Graph the following relations in the given interval:

a. $y = |x| + x + 1$ for $-2 \le x \le 2$ [Hint: Sketch by adding ordinates]

b. $y = |x| + |x-1|$ for $-2 \le x \le 3$

c. $y = 2^x + 2^{-x}$ for $-2 \le x \le 2$

d. $|x - y| = 1$ for $-1 \le x \le 3$.

10. Sketch the function $f(x) = |x^2 - 1| - 1$.

11. Given $y = f(x)$ as sketched below, sketch

a. $y = 2f(x)$

b. $y = -f(x)$

c. $y = f(-x)$

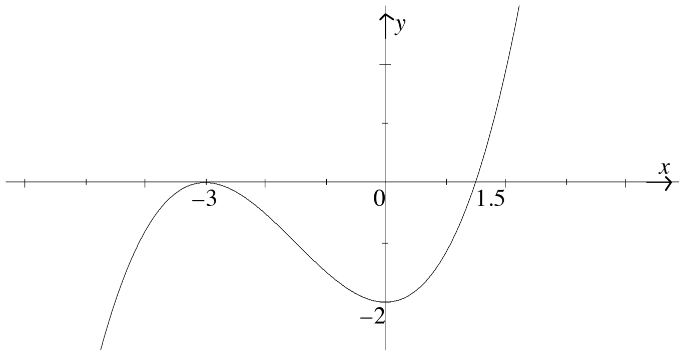
d. $y = f(x) + 4$

e. $y = f(x - 3)$

f. $y = f(x + 1) - 2$

g. $y = 3 - 2f(x - 3)$

h. $y = |f(x)|$



12. By sketching graphs solve the following equations:

a. $|2x| = 4$

b. $\frac{1}{x-2} = -1$

c. $x^3 = x^2$

d. $x^2 = \frac{1}{x}$

13. Solve $|x - 2| = 3$.

a. algebraically

b. geometrically.

14. The parabolas $y = (x - 1)^2$ and $y = (x - 3)^2$ intersect at a point P. Find the coordinates of P.

15. Sketch the circle $x^2 + y^2 - 2x - 14y + 25 = 0$. [Hint: Complete the squares.] Find the values of k, so that the line $y = k$ intersects the circle in two distinct points.

16. Solve $\frac{4}{5-x} = 1$, using a graph.

17. Find all real numbers x for which $|x - 2| = |x + 2|$.

18. Given that $Q(p) = p^2 - p$, find possible values of n if $Q(n) = 2$.

19. Solve $|x - 4| = 2x$.

a. algebraically

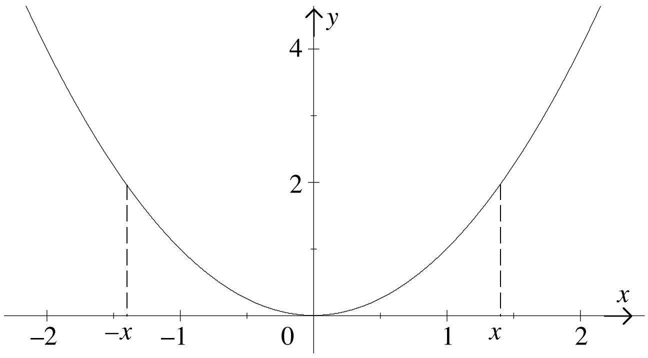
b. geometrically.

2.9 Even and odd functions

Definition:

> A function, $y = f(x)$, is *even* if $f(x) = f(-x)$ for all x in the domain of f.

Geometrically, an even function is symmetrical about the y-axis (it has line symmetry). The function $f(x) = x^2$ is an even function as $f(-x) = (-x)^2 = x^2 = f(x)$ for all values of x. We illustrate this on the following graph.

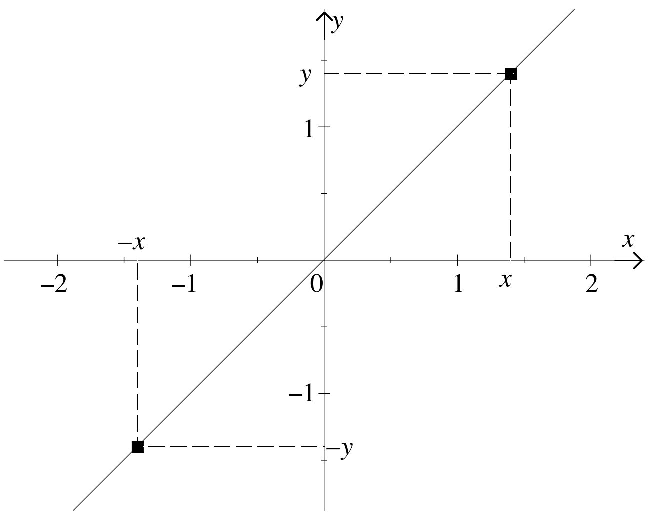


The graph of $y = x^2$.

Definition:

> A function, $y = f(x)$, is *odd* if $f(-x) = -f(x)$ for all x in the domain of f.

Geometrically, an odd function is symmetrical about the origin (it has rotational symmetry).
The function $f(x) = x$ is an odd function as $f(-x) = -x = -f(x)$ for all values of x. This is illustrated on the following graph.



The graph of $y = x$.

Example

Decide whether the following functions are even, odd or neither.

1. $f(x) = 3x^2 - 4$

2. $g(x) = \frac{1}{2x}$

3. $f(x) = x^3 - x^2$.

Solution

1.

$$f(-x) = 3(-x)^2 - 4 = 3x^2 - 4 = f(x)$$

The function $f(x) = 3x^2 - 4$ is even.

2.

$$g(-x) = \frac{1}{2(-x)} = \frac{1}{-2x} = -\frac{1}{2x} = -g(x)$$

Therefore, the function g is odd.

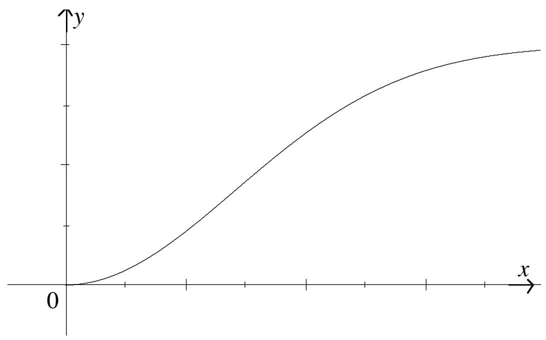
3.

$$f(-x) = (-x)^3 - (-x)^2 = -x^3 - x^2$$

This function is neither even (since $-x^3 - x^2 \neq x^3 - x^2$) nor odd (since $-x^3 - x^2 \neq -(x^3 - x^2)$).

Example

Sketched below is part of the graph of $y = f(x)$.

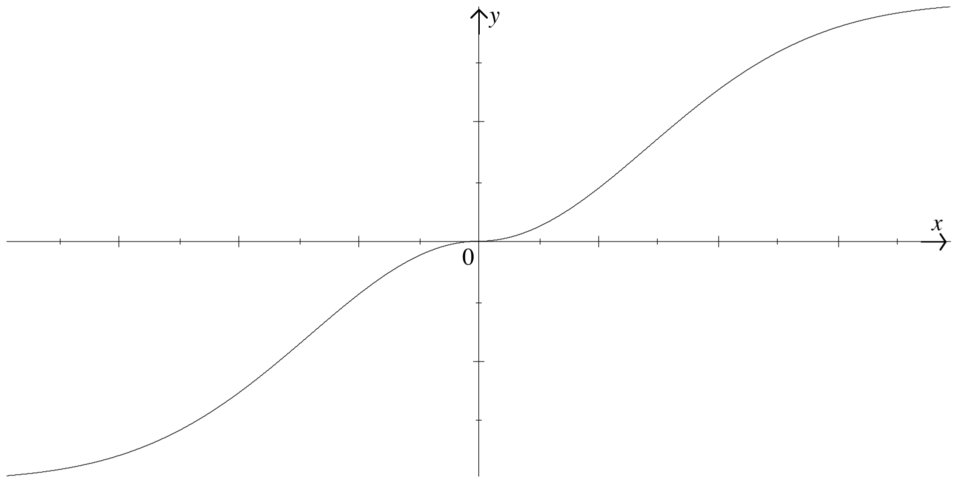


Complete the graph if $y = f(x)$ is

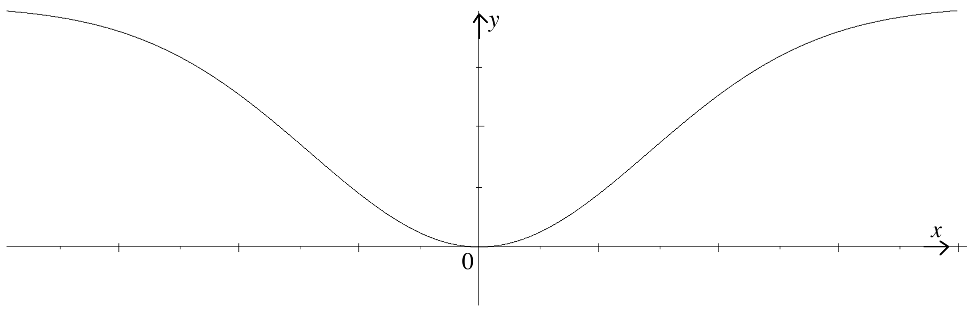
1. odd

2. even.

Solution



$y = f(x)$ is an odd function.



$y = f(x)$ is an even function.

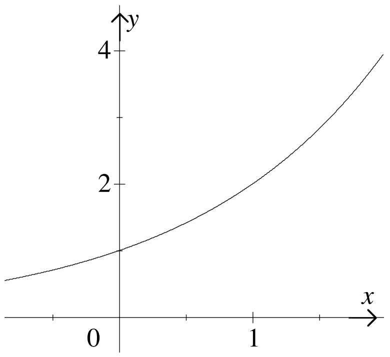
2.10 Increasing and decreasing functions

Here we will introduce the concepts of increasing and decreasing functions. In Chapter 5 we will relate these concepts to the derivative of a function.

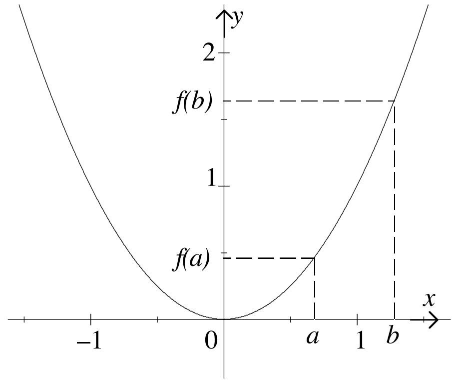
Definition:

> A function is *increasing* on an interval I, if for all a and b in I such that $a < b$, $f(a) < f(b)$.

The function $y = 2^x$ is an example of a function that is increasing over its domain. The function $y = x^2$ is increasing for all real $x > 0$.



The graph of $y = 2^x$. This function is increasing for all real x.



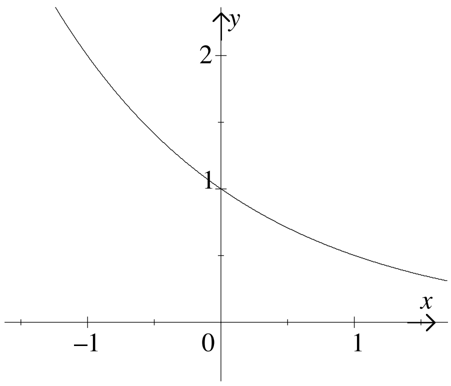
The graph of $y = x^2$. This function is increasing on the interval $x > 0$.

Notice that when a function is increasing it has a positive slope.

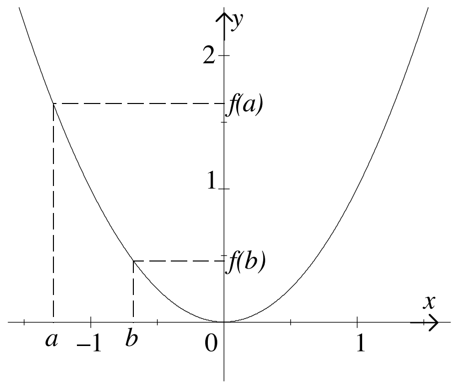
Definition:

> A graph is decreasing on an interval I, if for all a and b in I such that $a < b$, $f(a) > f(b)$.

The function $y = 2^{-x}$ is decreasing over its domain. The function $y = x^2$ is decreasing on the interval $x < 0$.



The graph of $y = 2^{-x}$. This function is decreasing for all real x.



The graph of $y = x^2$. This function is decreasing on the interval $x < 0$.

Notice that if a function is decreasing then it has negative slope.

2.11 Exercises

1. Given the graph below of $y = f(x)$:

a. State the domain and range.

b. Where is the graph

i increasing?

ii decreasing?

c. if k is a constant, find the values of k such that $f(x) = k$ has

i no solutions

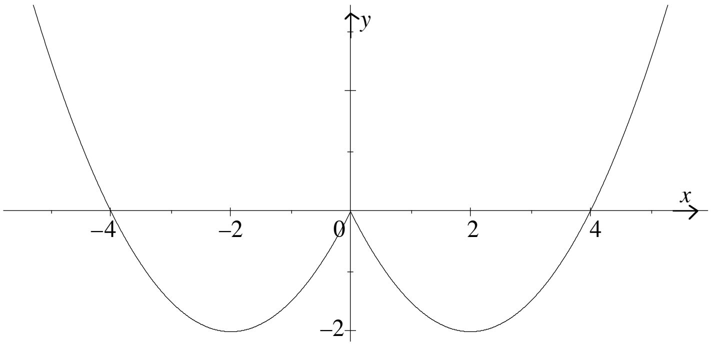
ii 1 solution

iii 2 solutions

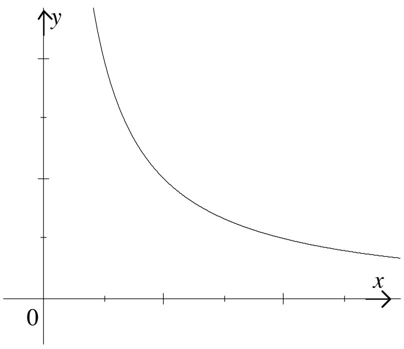
iv 3 solutions

v 4 solutions.

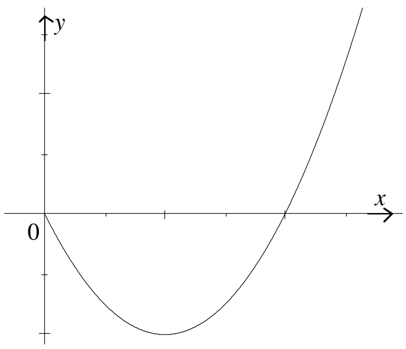
d. Is $y = f(x)$ even, odd or neither?



2. Complete the following functions if they are defined to be (a) even (b) odd.



$y = f(x)$



$y = g(x)$

3. Determine whether the following functions are odd, even or neither.

a. $y = x^4 + 2$ **b.** $y = \sqrt{4 - x^2}$ **c.** $y = 2^x$ **d.** $y = x^3 + 3x$

e. $y = \frac{x}{x^2}$ **f.** $y = \dfrac{1}{x^2 - 4}$ **g.** $y = \dfrac{1}{x^2 + 4}$ **h.** $y = \dfrac{x}{x^3 + 3}$

i. $y = 2^x + 2^{-x}$ **j.** $y = |x - 1| + |x + 1|$

4. Given $y = f(x)$ is even and $y = g(x)$ is odd, prove

a. if $h(x) = f(x) \cdot g(x)$ then $h(x)$ is odd

b. if $h(x) = (g(x))^2$ then $h(x)$ is even

c. if $h(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$, then $h(x)$ is odd

d. if $h(x) = f(x) \cdot (g(x))^2$ then $h(x)$ is even.

5. Consider the set of all odd functions which are defined at $x = 0$. Can you prove that for every odd function in this set $f(0) = 0$? If not, give a counter-example.

3 Piecewise functions and solving inequalities

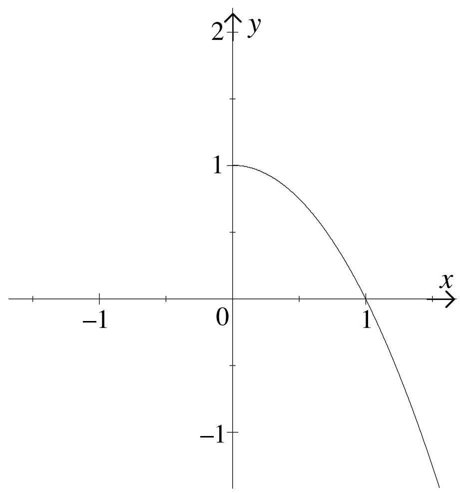
In this Chapter we will discuss functions that are defined piecewise (sometimes called piecemeal functions) and look at solving inequalities using both algebraic and graphical techniques.

3.1 Piecewise functions

3.1.1 Restricting the domain

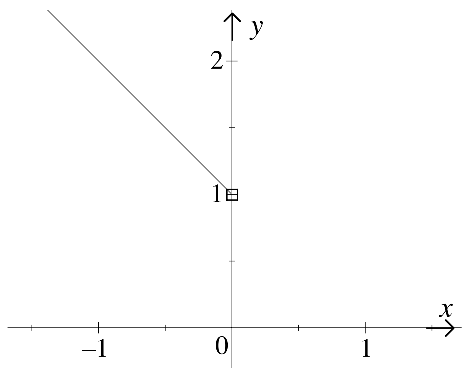
In Chapter 1 we saw how functions could be defined on a subinterval of their natural domain. This is frequently called *restricting* the domain of the function. In this Chapter we will extend this idea to define functions piecewise.

Sketch the graph of $y = 1 - x^2$ for $x \geq 0$.



The graph of $y = 1 - x^2$ for $x \geq 0$.

Sketch the graph of $y = 1 - x$ for $x < 0$.

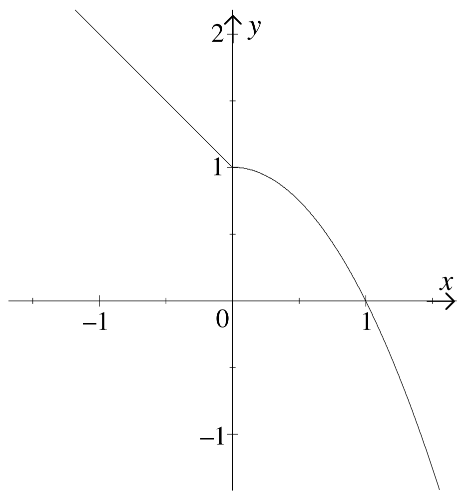


The graph of $y = 1 - x$ for $x < 0$.

We can now put these pieces together to define a function of the form

$$f(x) = \begin{cases} 1 - x^2 & \text{for } x \geq 0 \\ 1 - x & \text{for } x < 0 \end{cases}$$

We say that this function is defined *piecewise*. First note that it *is* a function; each value of x in the domain is assigned exactly one value of y. This is easy to see if we graph the function and use the vertical line test. We graph this function by graphing each piece of it in turn.



The graph shows that f defined in this way is a function. The two pieces of $y = f(x)$ meet so f is a continuous function.

The absolute value function

$$f(x) = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

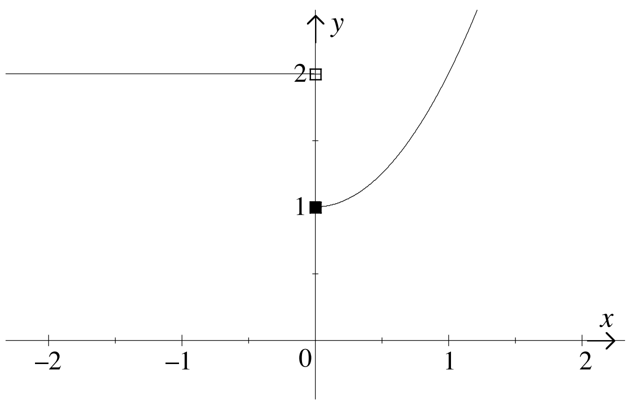
is another example of a piecewise function.

Example

Sketch the function

$$f(x) = \begin{cases} x^2 + 1 & \text{for } x \geq 0 \\ 2 & \text{for } x < 0 \end{cases}$$

Solution



This function is not continuous at $x = 0$ as the two branches of the graph do not meet.

Notice that we have put an open square (or circle) around the point $(0, 2)$ and a solid square (or circle) around the point $(0, 1)$. This is to make it absolutely clear that $f(0) = 1$ and not 2. When defining a function piecewise, we must be extremely careful to assign to each x exactly one value of y.

3.2 Exercises

1. For the function

$$f(x) = \begin{cases} 1 - x^2 & \text{for } x \geq 0 \\ 1 - x & \text{for } x < 0 \end{cases}$$

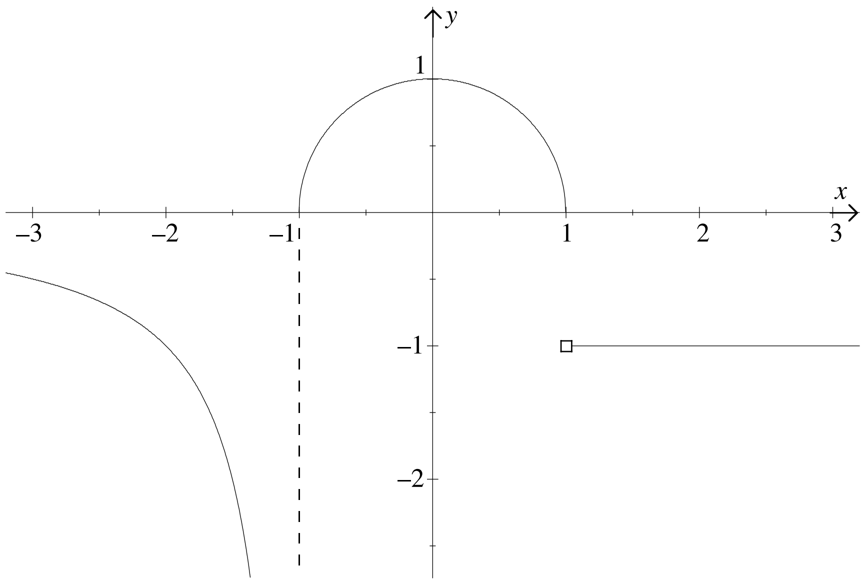
evaluate

a. $2f(-1) + f(2)$

b. $f(a^2)$

2. For the function given in **1**, solve $f(x) = 2$.

3. Below is the graph of $y = g(x)$. Write down the rules which define $g(x)$ given that its pieces are hyperbolic, circular and linear.



4. **a.** Sketch the graph of $y = f(x)$ if

$$f(x) = \begin{cases} -\sqrt{4 - x^2} & \text{for } -2 \leq x \leq 0 \\ x^2 - 4 & \text{for } x > 0 \end{cases}$$

b. State the range of f.

c. Solve

i $f(x) = 0$

ii $f(x) = -3$.

d. Find k if $f(x) = k$ has

i 0

ii 1

iii 2 solutions.

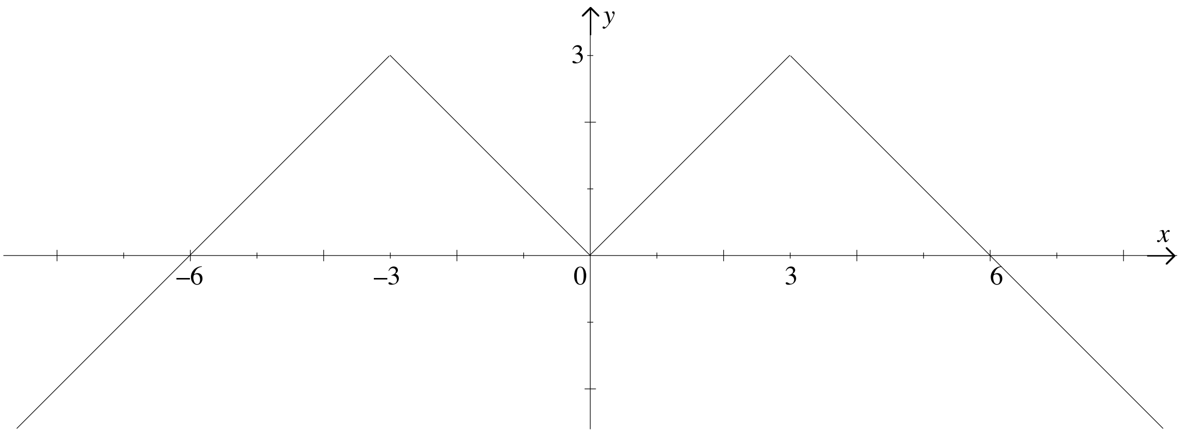
5. Sketch the graph of $y = f(x)$ if

$$f(x) = \begin{cases} 1 - |x - 1| & \text{for } x \geq 0 \\ |x + 1| & \text{for } x < 0 \end{cases}$$

6. Sketch the graph of $y = g(x)$ if

$$g(x) = \begin{cases} \frac{2}{x+2} & \text{for } x < -1 \\ 2 & \text{for } -1 \leq x < 1 \\ 2^x & \text{for } x \geq 1 \end{cases}$$

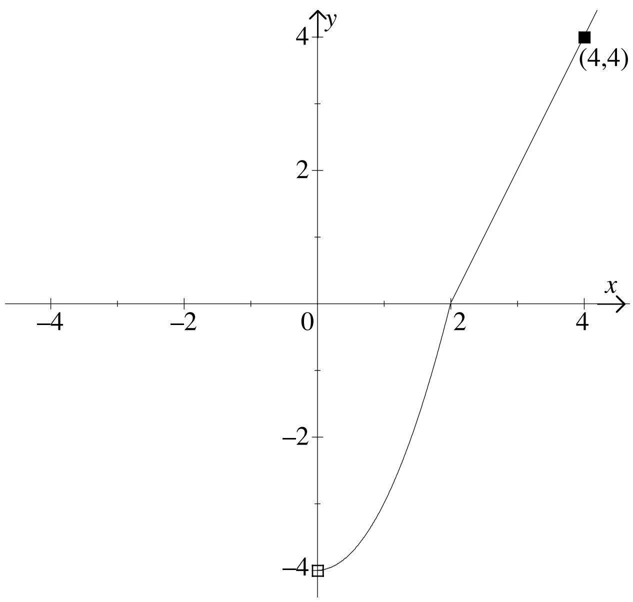
7. McMaths burgers are to modernise their logo as shown below.



Write down a piecewise function that represents this function using (a) 4 (b) 3 (c) 2 pieces (i.e. rules that define the function).

8. a. The following piecewise function is of the form

$$f(x) = \begin{cases} ax^2 + b & \text{for } 0 < x \leq 2 \\ cx + d & \text{for } x > 2 \end{cases}$$



Determine the values of a, b, c and d.

b. Complete the graph so that $f(x)$ is an odd function defined for all real x, $x \neq 0$.

c. Write down the equations that now define $f(x)$, $x \neq 0$.

3.3 Inequalities

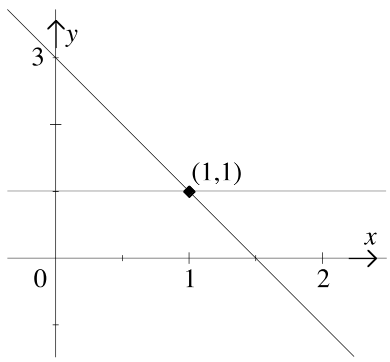
We can solve inequalities using both algebraic and graphical methods. Sometimes it is easier to use an algebraic method and sometimes a graphical one. For the following examples we will use both, as this allows us to make the connections between the algebra and the graphs.

Algebraic method

Graphical method

1. Solve $3 - 2x \geq 1$.
This is a (2 Unit) linear inequality. Remember to reverse the inequality sign when multiplying or dividing by a negative number.

$$\begin{aligned} 3 - 2x &\geq 1 \\ -2x &\geq -2 \\ x &\leq 1 \end{aligned}$$

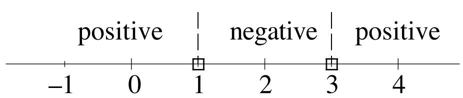


When is the line $y = 3 - 2x$ above or on the horizontal line $y = 1$? From the graph, we see that this is true for $x \leq 1$.

2. Solve $x^2 - 4x + 3 < 0$.
This is a (2 Unit) quadratic inequality. Factorise and use a number line.

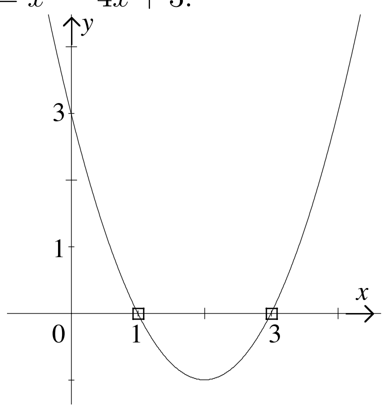
$$\begin{aligned} x^2 - 4x + 3 &< 0 \\ (x-3)(x-1) &< 0 \end{aligned}$$

The critical values are 1 and 3, which divide the number line into three intervals. We take points in each interval to determine the sign of the inequality; eg use $x = 0$, $x = 2$ and $x = 4$ as test values.



Thus, the solution is $1 < x < 3$.

Let $y = x^2 - 4x + 3$.

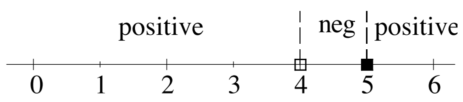


When does the parabola have negative y-values? OR When is the parabola under the x-axis? From the graph, we see that this happens when $1 < x < 3$.

3. Solve $\frac{1}{x-4} \le 1$.
This is a 3 Unit inequality. There is a variable in the denominator. Remember that a denominator can never be zero, so in this case $x \neq 4$. First multiply by the square of the denominator

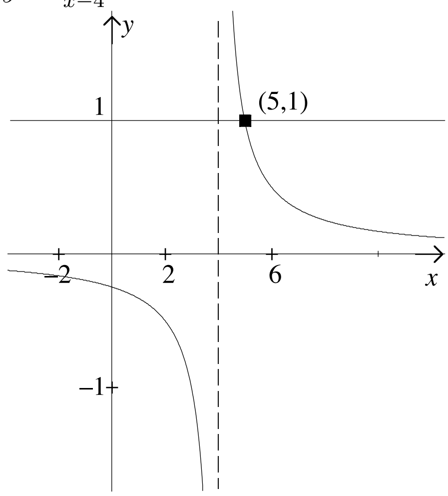
$$\begin{aligned} x-4 &\le (x-4)^2, x \neq 4 \\ x-4 &\le x^2-8x+16 \\ 0 &\le x^2-9x+20 \\ 0 &\le (x-4)(x-5) \end{aligned}$$

Mark the critical values on the number line and test $x = 0$, $x = 4.5$ and $x = 6$.



Therefore, $x < 4$ or $x \ge 5$.

Let $y = \frac{1}{x-4}$.



$y = \frac{1}{x-4}$ is not defined for $x = 4$. It is a hyperbola with vertical asymptote at $x = 4$. To solve our inequality we need to find the values of x for which the hyperbola lies on or under the line $y = 1$. $(5, 1)$ is the point of intersection. So, from the graph we see that $\frac{1}{x-4} \le 1$ when $x < 4$ or $x \ge 5$.

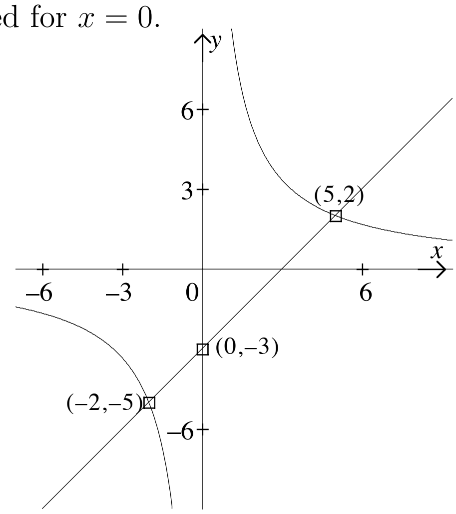
4. Solve $x - 3 < \frac{10}{x}$.
Consider $x - 3 = \frac{10}{x}, \quad x \neq 0$.
Multiply by x we get

$$\begin{aligned} x^2-3x &= 10 \\ x^2-3x-10 &= 0 \\ (x-5)(x+2) &= 0 \end{aligned}$$

Therefore, the critical values are -2, 0 and 5 which divide the number line into four intervals. We can use $x = -3$, $x = -1$, $x = 1$ and $x = 6$ as test values in the inequality. The points $x = -3$ and $x = 1$ satisfy the inequality, so the solution is $x < -2$ or $0 < x < 5$.

(Notice that we had to include 0 as one of our critical values.)

Sketch $y = x - 3$ and then $y = \frac{10}{x}$. Note that second of these functions is not defined for $x = 0$.



For what values of x does the line lie under the hyperbola? From the graph, we see that this happens when $x < -2$ or $0 < x < 5$.

Example

Sketch the graph of $y = |2x - 6|$.

Hence, where possible,

a. Solve

i $|2x - 6| = 2x$

ii $|2x - 6| > 2x$

iii $|2x - 6| = x + 3$

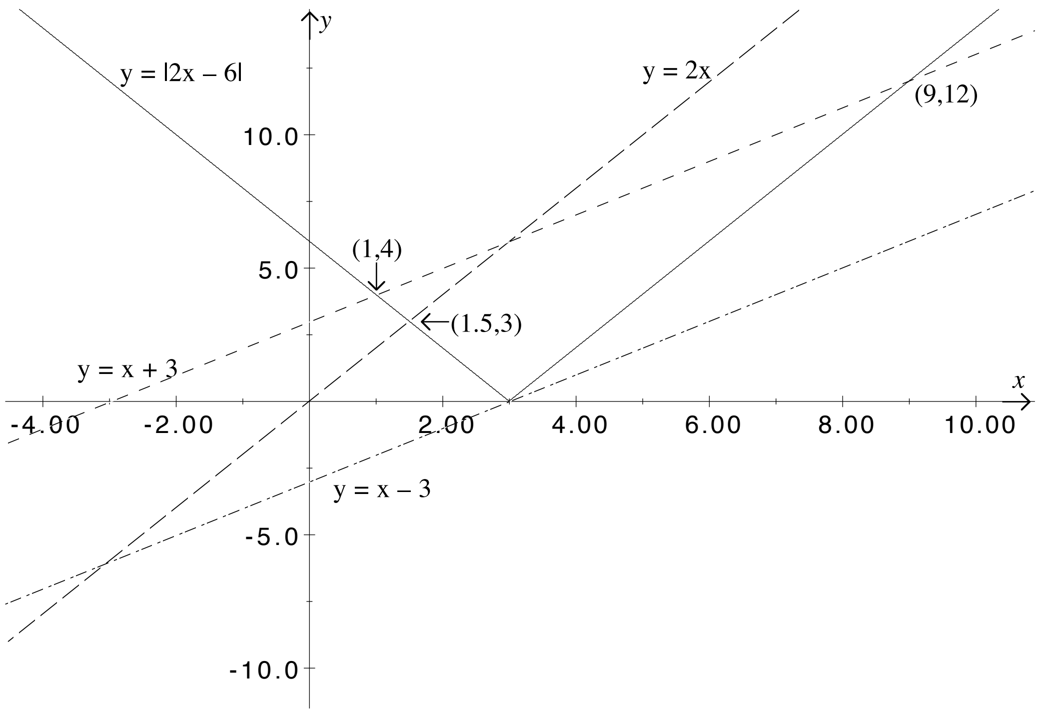
iv $|2x - 6| < x + 3$

v $|2x - 6| = x - 3$

b. Determine the values of k for which $|2x - 6| = x + k$ has exactly two solutions.

Solution

$$f(x) = |2x - 6| = \begin{cases} 2x - 6 & \text{for } x \geq 3 \\ -(2x - 6) & \text{for } x < 3 \end{cases}$$



a. i Mark in the graph of $y = 2x$. It is parallel to one arm of the absolute value graph. It has one point of intersection with $y = |2x - 6| = -2x + 6$ $(x < 3)$ at $x = 1.5$.

ii When is the absolute value graph above the line $y = 2x$? From the graph, when $x < 1.5$.

iii $y = x + 3$ intersects $y = |2x - 6|$ twice.
To solve $|2x - 6| = x + 3$, take $|2x - 6| = 2x - 6 = x + 3$ when $x \geq 3$. This gives us the solution $x = 9$. Then take $|2x - 6| = -2x + 6 = x + 3$ when $x < 3$ which gives us the solution $x = 1$.

iv When is the absolute value graph below the line $y = x + 3$?
From the graph, $1 < x < 9$.

v $y = x - 3$ intersects the absolute value graph at $x = 3$ only.

b. k represents the y-intercept of the line $y = x + k$. When $k = -3$, there is one point of intersection. (See (a) (v) above). For $k > -3$, lines of the form $y = x + k$ will have two points of intersection. Hence $|2x - 6| = x + k$ will have two solutions for $k > -3$.

3.4 Exercises

1. Solve

a. $x^2 \leq 4x$

b. $\frac{4p}{p+3} \leq 1$

c. $\frac{7}{9-x^2} > -1$

2. **a.** Sketch the graph of $y = 4x(x - 3)$.

b. Hence solve $4x(x - 3) \leq 0$.

3. **a.** Find the points of intersection of the graphs $y = 5 - x$ and $y = \frac{4}{x}$.

b. On the same set of axes, sketch the graphs of $y = 5 - x$ and $y = \frac{4}{x}$.

c. Using part (ii), or otherwise, write down all the values of x for which

$$5 - x > \frac{4}{x}$$

4. **a.** Sketch the graph of $y = 2^x$.

b. Solve $2^x < \frac{1}{2}$.

c. Suppose $0 < a < b$ and consider the points $\text{A}(a, 2^a)$ and $\text{B}(b, 2^b)$ on the graph of $y = 2^x$. Find the coordinates of the midpoint M of the segment AB.
Explain why

$$\frac{2^a + 2^b}{2} > 2^{\frac{a+b}{2}}$$

5. **a.** Sketch the graphs of $y = x$ and $y = |x - 5|$ on the same diagram.

b. Solve $|x - 5| > x$.

c. For what values of m does $mx = |x - 5|$ have exactly

i two solutions

ii no solutions

6. Solve $5x^2 - 6x - 3 \leq |8x|$.

4 Polynomials

Many of the functions we have been using so far have been polynomials. In this Chapter we will study them in more detail.

Definition

A real *polynomial*, $P(x)$, of degree n is an expression of the form

$$P(x) = p_n x^n + p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \cdots + p_2x^2 + p_1x + p_0$$

where $p_n \neq 0$, p_0, p_1, $\cdots$, p_n are real and n is an integer ≥ 0.

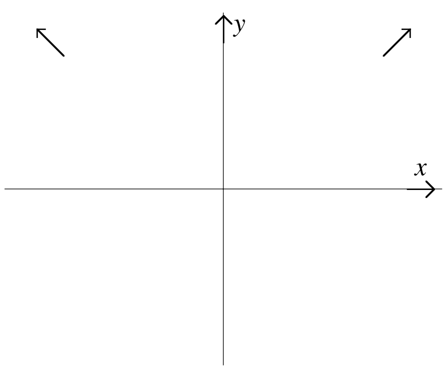
All polynomials are defined for all real x and are continuous functions.
We are familiar with the quadratic polynomial, $Q(x) = ax^2 + bx + c$ where $a \neq 0$. This polynomial has degree 2.
The function $f(x) = \sqrt{x} + x$ is not a polynomial as it has a power which is not an integer ≥ 0 and so does not satisfy the definition.

4.1 Graphs of polynomials and their zeros

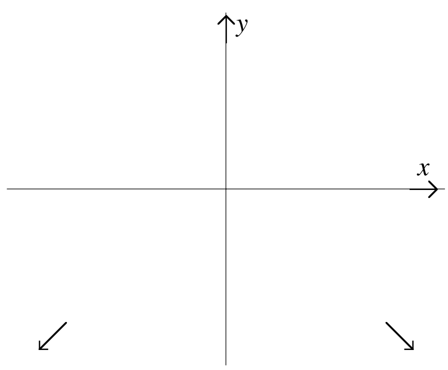
4.1.1 Behaviour of polynomials when $|x|$ is large

One piece of information that can be a great help when sketching a polynomial is the way it behaves for values of x when $|x|$ is large. That is, values of x which are large in magnitude.
The term of the polynomial with the highest power of x is called the *leading* or *dominant* term. For example, in the polynomial $P(x) = x^6 - 3x^4 - 1$, the term x^6 is the dominant term.
When $|x|$ is large, the dominant term determines how the graph behaves as it is so much larger in magnitude than all the other terms.

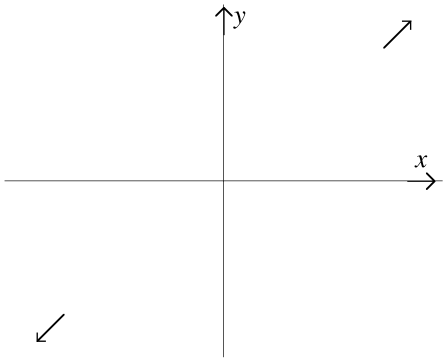
How the graph behaves for $|x|$ large depends on the *power* and *coefficient* of the dominant term.
There are four possibilities which we summarise in the following diagrams:



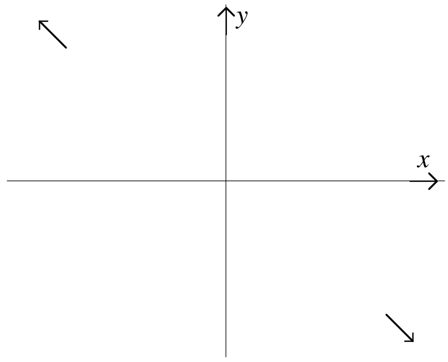
1. Dominant term with even power and positive coefficient, eg $y = x^2$.



2. Dominant term with even power and negative coefficient, eg $Q(x) = -x^2$.



3. Dominant term with odd power and positive coefficient, eg $y = x^3$.



4. Dominant term with odd power and negative coefficient, eg $Q(x) = -x^3$.

This gives us a good start to graphing polynomials. All we need do now is work out what happens in the middle. In Chapter 5 we will use calculus methods to do this. Here we will use our knowledge of the roots of polynomials to help complete the picture.

4.1.2 Polynomial equations and their roots

If, for a polynomial $P(x)$, $P(k) = 0$ then we can say

1. $x = k$ is a root of the equation $P(x) = 0$.

2. $x = k$ is a zero of $P(x)$.

3. k is an x-intercept of the graph of $P(x)$.

4.1.3 Zeros of the quadratic polynomial

The quadratic polynomial equation $Q(x) = ax^2 + bx + c = 0$ has two roots that may be:

1. real (rational or irrational) and distinct,

2. real (rational or irrational) and equal,

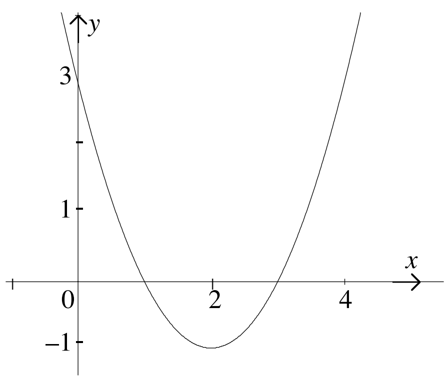
3. complex (not real).

We will illustrate all of these cases with examples, and will show the relationship between the nature and number of zeros of $Q(x)$ and the x-intercepts (if any) on the graph.

1. Let $Q(x) = x^2 - 4x + 3$.
We find the zeros of $Q(x)$ by solving the equation $Q(x) = 0$.

$$\begin{aligned} x^2 - 4x + 3 &= 0 \\ (x-1)(x-3) &= 0 \\ \text{Therefore } x &= 1 \text{ or } 3. \end{aligned}$$

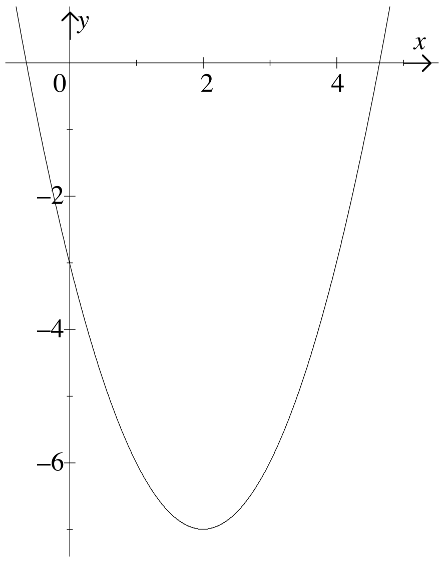
The roots are rational (hence real) and distinct.



2. Let $Q(x) = x^2 - 4x - 3$.
Solving the equation $Q(x) = 0$ we get,

$$\begin{aligned} x^2 - 4x - 3 &= 0 \\ x &= \frac{4 \pm \sqrt{16 + 12}}{2} \\ \text{Therefore } x &= 2 \pm \sqrt{7}. \end{aligned}$$

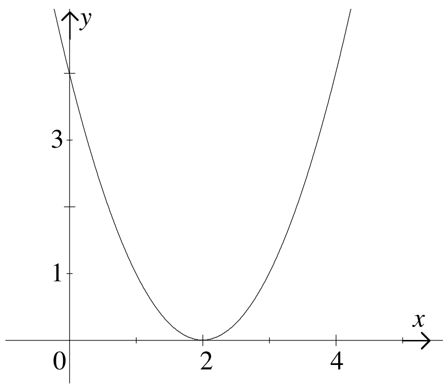
The roots are irrational (hence real) and distinct.



3. Let $Q(x) = x^2 - 4x + 4$.
Solving the equation $Q(x) = 0$ we get,

$$\begin{aligned} x^2 - 4x + 4 &= 0 \\ (x - 2)^2 &= 0 \\ \text{Therefore } x &= 2. \end{aligned}$$

The roots are rational (hence real) and equal. $Q(x) = 0$ has a repeated or double root at $x = 2$.

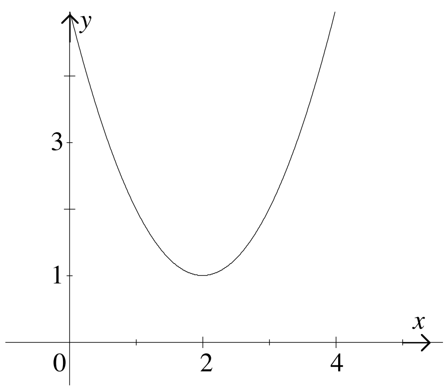


Notice that the graph turns at the double root $x = 2$.

4. Let $Q(x) = x^2 - 4x + 5$.
Solving the equation $Q(x) = 0$ we get,

$$\begin{aligned} x^2 - 4x + 5 &= 0 \\ x &= \frac{4 \pm \sqrt{16 - 20}}{2} \\ \text{Therefore } x &= 2 \pm \sqrt{-4}. \end{aligned}$$

There are no real roots. In this case the roots are complex.



Notice that the graph does not intersect the x-axis. That is $Q(x) > 0$ for all real x. Therefore Q is positive definite.

We have given above four examples of quadratic polynomials to illustrate the relationship between the zeros of the polynomials and their graphs.

In particular we saw that:

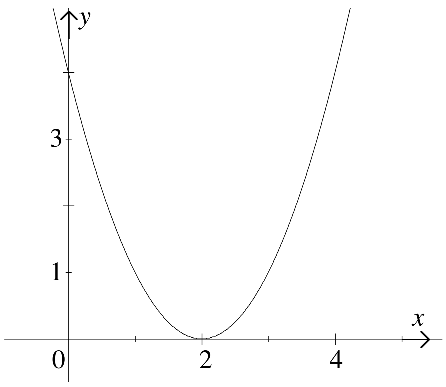
i if the quadratic polynomial has two real distinct zeros, then the graph of the polynomial cuts the x-axis at two distinct points;

ii if the quadratic polynomial has a real double (or repeated) zero, then the graph sits on the x-axis;

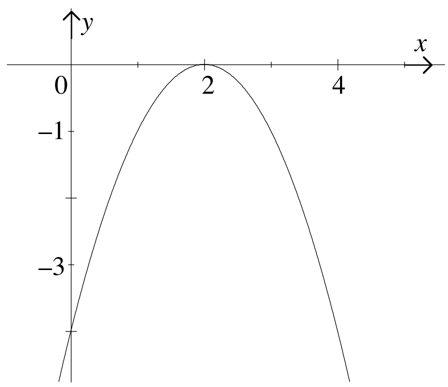
iii if the quadratic polynomial has no real zeros, then the graph does not intersect the x-axis at all.

So far, we have only considered quadratic polynomials where the coefficient of the x^2 term is positive which gives us a graph which is *concave up*. If we consider polynomials $Q(x) = ax^2 + bx + c$ where $a < 0$ then we will have a graph which is *concave down*.

For example, the graph of $Q(x) = -(x^2 - 4x + 4)$ is the reflection in the x-axis of the graph of $Q(x) = x^2 - 4x + 4$. (See Chapter 2.)



The graph of $Q(x) = x^2 - 4x + 4$.



The graph of $Q(x) = -(x^2 - 4x + 4)$.

4.1.4 Zeros of cubic polynomials

A real cubic polynomial has an equation of the form

$$P(x) = ax^3 + bx^2 + cx + d$$

where $a \neq 0$, a, b, c and d are real. It has 3 zeros which may be:

i 3 real distinct zeros;

ii 3 real zeros, all of which are equal (3 equal zeros);

iii 3 real zeros, 2 of which are equal;

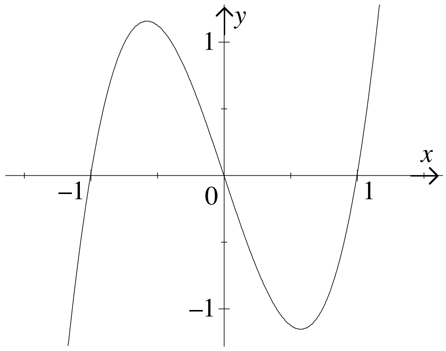
iv 1 real zero and 2 complex zeros.

We will illustrate these cases with the following examples:

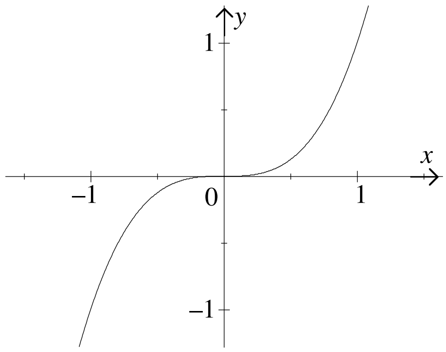
1. Let $Q(x) = 3x^3 - 3x$.
Solving the equation $Q(x) = 0$ we get:.

$$\begin{aligned} 3x^3 - 3x &= 0 \\ 3x(x-1)(x+1) &= 0 \\ \text{Therefore } x &= -1 \text{ or } 0 \text{ or } 1 \end{aligned}$$

The roots are real (in fact rational) and distinct.



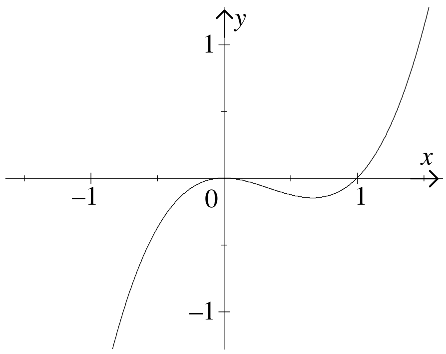
2. Let $Q(x) = x^3$.
Solving $Q(x) = 0$ we get that $x^3 = 0$.
We can write this as $(x-0)^3 = 0$.
So, this equation has three equal real roots at $x = 0$.



3. Let $Q(x) = x^3 - x^2$.
Solving the equation $Q(x) = 0$ we get,

$$\begin{aligned} x^3 - x^2 &= 0 \\ x^2(x-1) &= 0 \\ \text{Therefore } x &= 0 \text{ or } 1. \end{aligned}$$

The roots are real with a double root at $x = 0$ and a single root at $x = 1$.

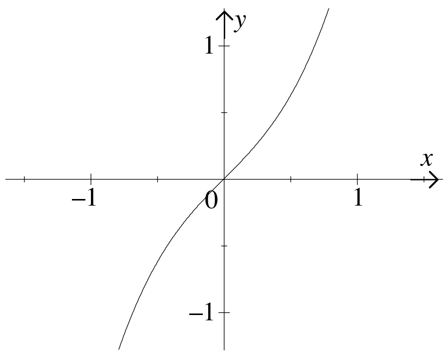


The graph turns at the double root.

4. Let $Q(x) = x^3 + x$.
Solving the equation $Q(x) = 0$ we get,

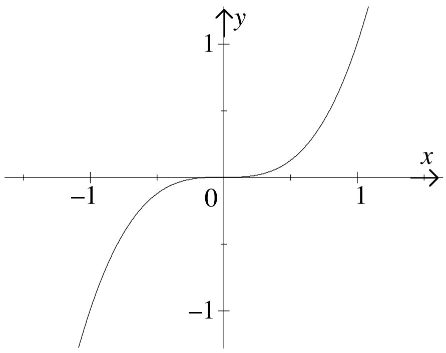
$$\begin{aligned} x^3 + x &= 0 \\ x(x^2+1) &= 0 \\ \text{Therefore } x &= 0. \end{aligned}$$

There is one real root at $x = 0$.
$x^2 + 1 = 0$ does not have any real solutions.

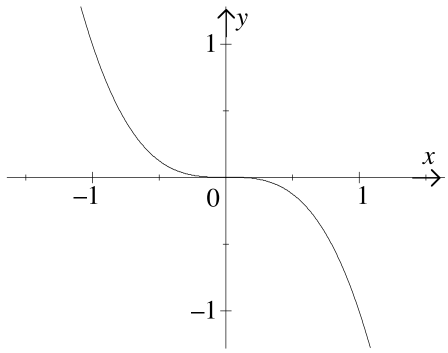


The graph intersects the x-axis once only.

Again, in the above examples we have looked only at cubic polynomials where the coefficient of the x^3 term is positive. If we consider the polynomial $P(x) = -x^3$ then the graph of this polynomial is the reflection of the graph of $P(x) = x^3$ in the x-axis.



The graph of $Q(x) = x^3$.



The graph of $Q(x) = -x^3$.

4.2 Polynomials of higher degree

We will write down a few rules that we can use when we have a polynomial of degree ≥ 3.

If $P(x)$ is a real polynomial of degree n then:

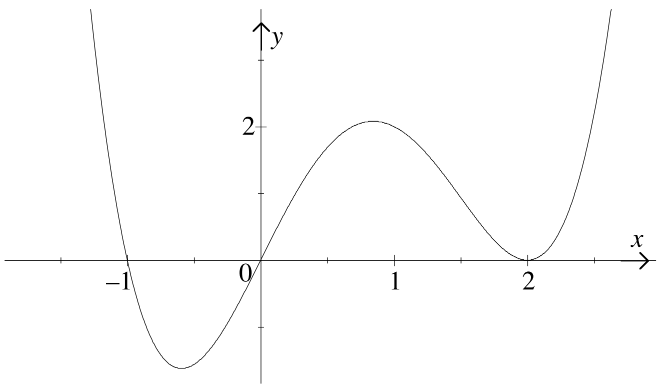
1. $P(x) = 0$ has at most n real roots;

2. if $P(x) = 0$ has a repeated root with an even power then the graph of $P(x)$ turns at this repeated root;

3. if $P(x) = 0$ has a repeated root with an odd power then the graph of $P(x)$ has a horizontal point of inflection at this repeated root.

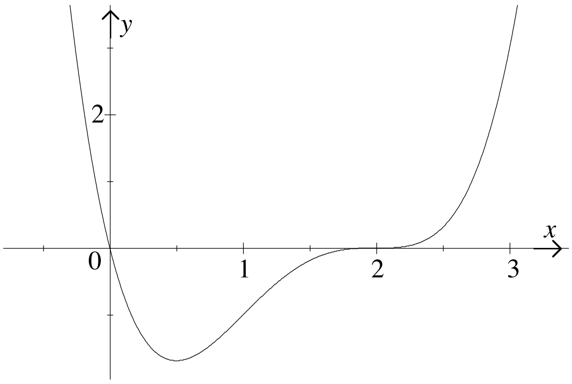
For example, **1.** tells us that if we have a quartic polynomial equation $f(x) = 0$. Then we know that $f(x) = 0$ has ≤ 4 real roots.

We can illustrate **2.** by the sketching $f(x) = x(x-2)^2(x+1)$. Notice how the graph sits on the x-axis at $x = 2$.



The graph of $f(x) = x(x+1)(x-2)^2$.

We illustrate **3.** by sketching the graph of $f(x) = x(x-2)^3$. Notice the horizontal point of inflection at $x = 2$.



The graph of $f(x) = x(x-2)^3$.

4.3 Exercises

1. Sketch the graphs of the following polynomials if $y = P(x)$ is:

a. $x(x+1)(x-3)$

b. $x(x+1)(3-x)$

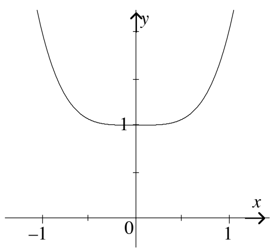
c. $(x+1)^2(x-3)$

d. $(x+1)(x^2-4x+5)$

2. The graphs of the following quartic polynomials are sketched below. Match the graph with the polynomial.

a. $y = x^4$ **b.** $y = x^4 - 1$ **c.** $y = x^4 + 1$ **d.** $y = 1 - x^4$ **e.** $y = (x-1)^4$ **f.** $y = (x+1)^4$

i



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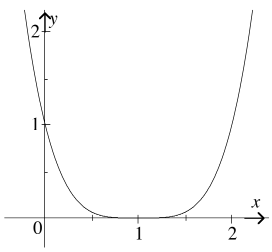

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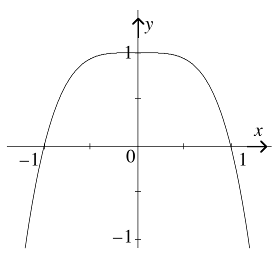

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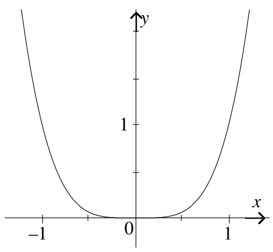

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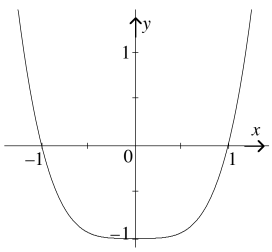

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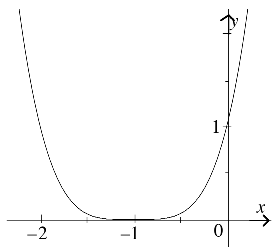


3. Sketch the graphs of the following quartic polynomials if $y = C(x)$ is:

a. $x(x-1)(x+2)(x+3)$

b. $x(x-1)(x+2)(3-x)$

c. $x^2(x-1)(x-3)$

d. $(x+1)^2(x-3)^2$

e. $(x+1)^3(x-3)$

f. $(x+1)^3(3-x)$

g. $x(x+1)(x^2-4x+5)$

h. $x^2(x^2-4x+5)$.

4. By sketching the appropriate polynomial, solve:

a. $x^2 - 4x - 12 < 0$

b. $(x+2)(x-3)(5-x) > 0$

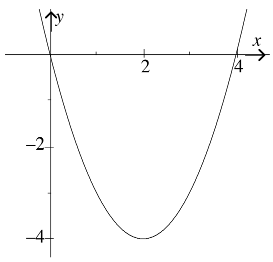
c. $(x+2)^2(5-x) > 0$

d. $(x+2)^3(5-x) \geq 0$.

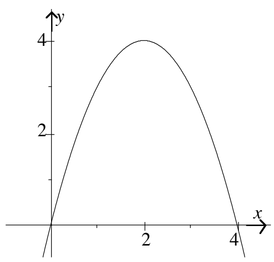
5. For what values of k will $P(x) \geq 0$ for all real x if $P(x) = x^2 - 4x - 12 + k$?

6. The diagrams show the graph of $y = P(x)$ where $P(x) = a(x-b)(x-c)^d$.
In each case determine possible values for a, b, c and d.

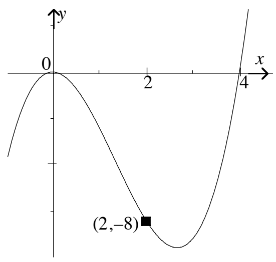
a.



b.



c.



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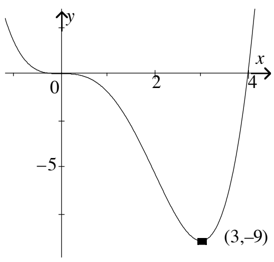

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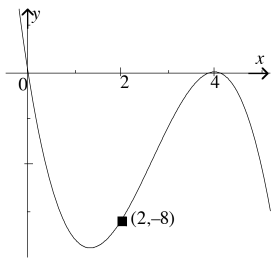
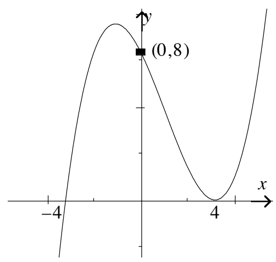


f.



7. The graph of the polynomial $y = f(x)$ is given below. It has a local maximum and minimum as marked. Use the graph to answer the following questions.

a. State the roots of $f(x) = 0$.

b. What is the value of the repeated root.

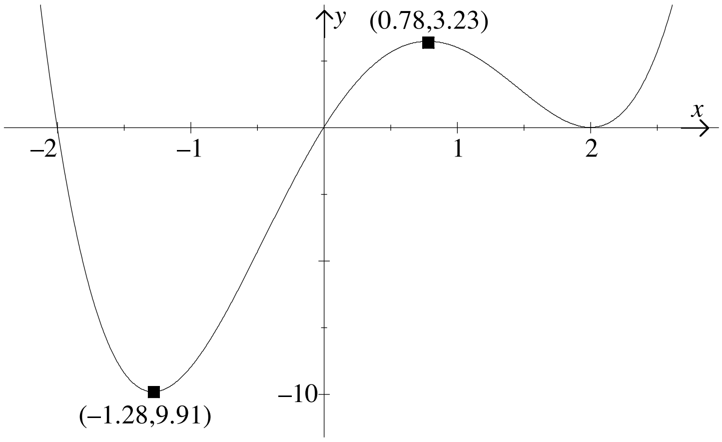
c. For what values of k does the equation $f(x) = k$ have exactly 3 solutions.

d. Solve the inequality $f(x) < 0$.

e. What is the *least* possible degree of $f(x)$?

f. State the value of the constant of $f(x)$.

g. For what values of k is $f(x) + k \geq 0$ for all real x.



The graph of the polynomial $y = f(x)$

4.4 Factorising polynomials

So far for the most part, we have looked at polynomials which were already factorised. In this section we will look at methods which will help us factorise polynomials with degree > 2.

4.4.1 Dividing polynomials

Suppose we have two polynomials $P(x)$ and $A(x)$, with the degree of $P(x) \geq$ the degree of $A(x)$, and $P(x)$ is divided by $A(x)$. Then

$$\frac{P(x)}{A(x)} = Q(x) + \frac{R(x)}{A(x)},$$

where $Q(x)$ is a polynomial called the *quotient* and $R(x)$ is a polynomial called the *remainder*, with the degree of $R(x) <$ degree of $A(x)$.

We can rewrite this as

$$P(x) = A(x) \cdot Q(x) + R(x).$$

For example: If $P(x) = 2x^3 + 4x + 3$ and $A(x) = x - 2$, then $P(x)$ can be divided by $A(x)$ as follows:

$$\begin{array}{r|l}
 & 2x^2 + 4x + 12 \\ \hline
x - 2 & 2x^3 + 0x^2 + 4x - 3 \\
 & \underline{2x^3 - 4x^2} \\
 & \quad 4x^2 + 4x - 3 \\
 & \quad \underline{4x^2 - 8x} \\
 & \qquad 12x - 3 \\
 & \qquad \underline{12x - 24} \\
 & \qquad\qquad 21
\end{array}$$

The quotient is $2x^2 + 4x + 12$ and the remainder is 21. We have

$$\frac{2x^3 + 4x + 3}{x - 2} = 2x^2 + 4x + 12 + \frac{21}{x - 2}.$$

This can be written as

$$2x^3 + 4x - 3 = (x - 2)(2x^2 + 4x + 12) + 21.$$

Note that the degree of the "polynomial" 21 is 0.

4.4.2 The Remainder Theorem

If the polynomial $f(x)$ is divided by $(x - a)$ then the remainder is $f(a)$.

Proof:

Following the above, we can write

$$f(x) = A(x) \cdot Q(x) + R(x),$$

where $A(x) = (x - a)$. Since the degree of $A(x)$ is 1, the degree of $R(x)$ is zero. That is, $R(x) = r$ where r is a constant.

$$\begin{aligned} f(x) &= (x-a)Q(x) + r \quad \text{where } r \text{ is a constant.} \\ f(a) &= 0 \cdot Q(a) + r \\ &= r \end{aligned}$$

So, if $f(x)$ is divided by $(x - a)$ then the remainder is $f(a)$.

Example

Find the remainder when $P(x) = 3x^4 - x^3 + 30x - 1$ is divided by **a.** $x + 1$, **b.** $2x - 1$.

Solution

a. Using the Remainder Theorem:

$$\begin{aligned} \text{Remainder} &= P(-1) \\ &= 3 - (-1) - 30 - 1 \\ &= -27 \end{aligned}$$

b.

$$\begin{aligned} \text{Remainder} &= P(\frac{1}{2}) \\ &= 3(\frac{1}{2})^4 - (\frac{1}{2})^3 + 30(\frac{1}{2}) - 1 \\ &= \frac{3}{16} - \frac{1}{8} + 15 - 1 \\ &= 14\frac{1}{16} \end{aligned}$$

Example

When the polynomial $f(x)$ is divided by $x^2 - 4$, the remainder is $5x + 6$. What is the remainder when $f(x)$ is divided by $(x - 2)$?

Solution

Write $f(x) = (x^2 - 4) \cdot q(x) + (5x + 6)$. Then

$$\begin{aligned} \text{Remainder} &= f(2) \\ &= 0 \cdot q(2) + 16 \\ &= 16 \end{aligned}$$

A consequence of the Remainder Theorem is the Factor Theorem which we state below.

4.4.3 The Factor Theorem

If $x = a$ is a zero of $f(x)$, that is $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$ and $f(x)$ may be written as

$$f(x) = (x - a)q(x)$$

for some polynomial $q(x)$.

Also, if $(x - a)$ and $(x - b)$ are factors of $f(x)$ then $(x - a)(x - b)$ is a factor of $f(x)$ and

$$f(x) = (x - a)(x - b) \cdot Q(x)$$

for some polynomial $Q(x)$.

Another useful fact about zeros of polynomials is given below for a polynomial of degree 3.

If a (real) polynomial

$$P(x) = ax^3 + bx^2 + cx + d,$$

where $a \neq 0$, a, b, c and d are real, has exactly 3 real zeros α, β and γ, then

$$P(x) = a(x - \alpha)(x - \beta)(x - \gamma) \qquad (1)$$

Furthermore, by expanding the right hand side of (1) and equating coefficients we get:

i

$$\alpha + \beta + \gamma = -\frac{b}{a};$$

ii

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a};$$

iii

$$\alpha\beta\gamma = -\frac{d}{a}.$$

This result can be extended for polynomials of degree n. We will give the partial result for $n = 4$.

If

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e$$

is a polynomial of degree 4 with real coefficents, and $P(x)$ has four real zeros α, β, γ and δ, then

$$P(x) = a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$$

and expanding and equating as above gives

$$\alpha\beta\gamma\delta = \frac{e}{a}.$$

If $a = 1$ and the equation $P(x) = 0$ has a root which is an integer, then that integer must be a factor of the constant term. This gives us a place to start when looking for factors of a polynomial. That is, we look at all the factors of the constant term to see which ones (if any) are roots of the equation $P(x) = 0$.

Example

Let $f(x) = 4x^3 - 8x^2 - x + 2$

a. Factorise $f(x)$.

b. Sketch the graph of $y = f(x)$.

c. Solve $f(x) \geq 0$.

Solution

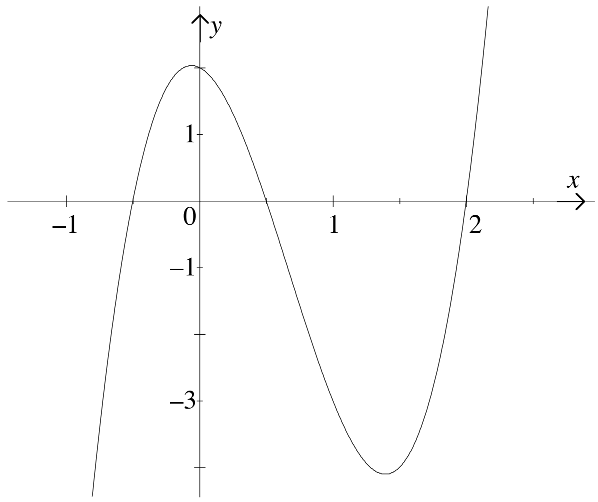
a. Consider the factors of the constant term, 2. We check to see if ± 1 and ± 2 are solutions of the equation $f(x) = 0$ by substitution. Since $f(2) = 0$, we know that $(x - 2)$ is a factor of $f(x)$. We use long division to determine the quotient.

$$\begin{array}{r|l} & 4x^2 - 1 \\ \hline x - 2 & 4x^3 - 8x^2 - x + 2 \\ & \underline{4x^3 - 8x^2} \\ & \qquad\qquad - x + 2 \\ & \qquad\qquad \underline{- x + 2} \end{array}$$

So,

$$\begin{aligned} f(x) &= (x - 2)(4x^2 - 1) \\ &= (x - 2)(2x - 1)(2x + 1) \end{aligned}$$

b.



The graph of $f(x) = 4x^3 - 8x^2 - x + 2$.

c. $f(x) \geq 0$ when $-\frac{1}{2} \leq x \leq \frac{1}{2}$ or $x \geq 2$.

Example

Show that $(x-2)$ and $(x-3)$ are factors of $P(x) = x^3 - 19x + 30$, and hence solve $x^3 - 19x + 30 = 0$.

Solution

$P(2) = 8 - 38 + 30 = 0$ and $P(3) = 27 - 57 + 30 = 0$ so $(x-2)$ and $(x-3)$ are both factors of $P(x)$ and $(x-2)(x-3) = x^2 - 5x + 6$ is also a factor of $P(x)$. Long division of $P(x)$ by $x^2 - 5x + 6$ gives a quotient of $(x+5)$.

So,

$$P(x) = x^3 - 19x + 30 = (x-2)(x-3)(x+5).$$

Solving $P(x) = 0$ we get $(x-2)(x-3)(x+5) = 0$.

That is, $x = 2$ or $x = 3$ or $x = -5$.

Instead of using long division we could have used the facts that

i the polynomial cannot have more than three real zeros;

ii the product of the zeros must be equal to -30.

Let α be the unknown root.

Then $2 \cdot 3 \cdot \alpha = -30$, so that $\alpha = -5$. Therefore the solution of $P(x) = x^3 - 19x + 30 = 0$ is $x = 2$ or $x = 3$ or $x = -5$.

4.5 Exercises

1. When the polynomial $P(x)$ is divided by $(x-a)(x-b)$ the quotient is $Q(x)$ and the remainder is $R(x)$.

a. Explain why $R(x)$ is of the form $mx+c$ where m and c are constants.

b. When a polynomial is divided by $(x-2)$ and $(x-3)$, the remainders are 4 and 9 respectively. Find the remainder when the polynomial is divided by x^2-5x+6.

c. When $P(x)$ is divided by $(x-a)$ the remainder is a^2. Also, $P(b)=b^2$. Find $R(x)$ when $P(x)$ is divided by $(x-a)(x-b)$.

2. **a.** Divide the polynomial $f(x)=2x^4+13x^3+18x^2+x-4$ by $g(x)=x^2+5x+2$. Hence write $f(x)=g(x)q(x)+r(x)$ where $q(x)$ and $r(x)$ are polynomials.

b. Show that $f(x)$ and $g(x)$ have no common zeros. (Hint: Assume that α is a common zero and show by contradiction that α does not exist.)

3. For the following polynomials,

i factorise

ii solve $P(x)=0$

iii sketch the graph of $y=P(x)$.

a. $P(x)=x^3-x^2-10x-8$

b. $P(x)=x^3-x^2-16x-20$

c. $P(x)=x^3+4x^2-8$

d. $P(x)=x^3-x^2+x-6$

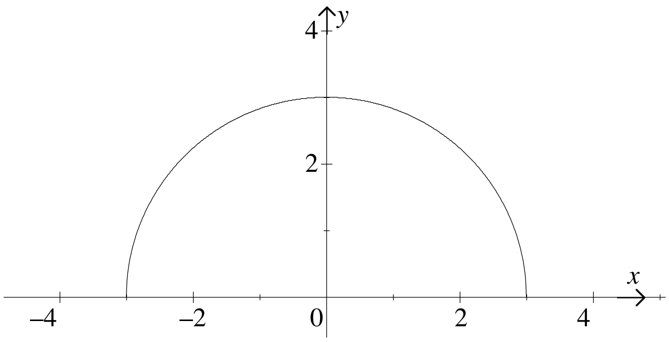
e. $P(x)=2x^3-3x^2-11x+6$

5 Solutions to exercises

1.4 Solutions

1. a. The domain of $f(x) = \sqrt{9 - x^2}$ is all real x where $-3 \leq x \leq 3$. The range is all real y such that $0 \leq y \leq 3$.

b.

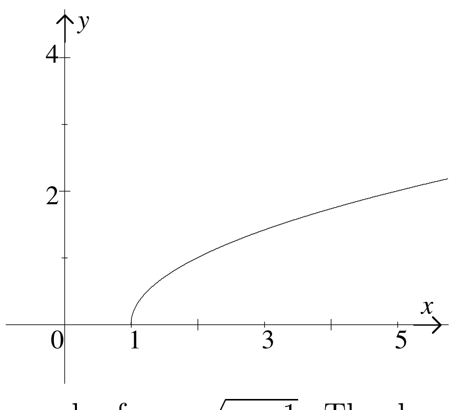


The graph of $f(x) = \sqrt{9 - x^2}$.

2.

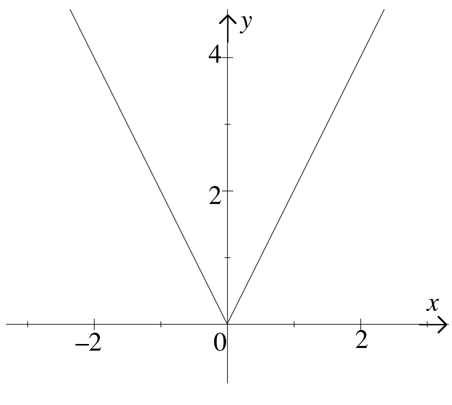
$$
\begin{aligned}
\frac{\psi(x+h) - \psi(x)}{h} &= \frac{(x+h)^2 + 5 - (x^2 + 5)}{h} \\
&= \frac{x^2 + 2xh + h^2 + 5 - x^2 - 5}{h} \\
&= \frac{h^2 + 2xh}{h} \\
&= h + 2x
\end{aligned}
$$

3. a.



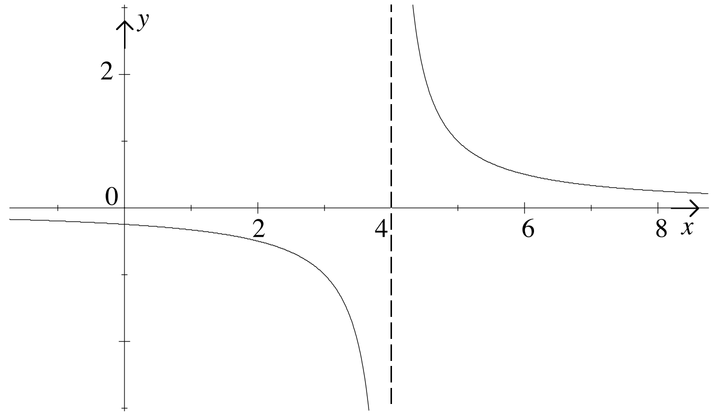
The graph of $y = \sqrt{x - 1}$. The domain is all real $x \geq 1$ and the range is all real $y \geq 0$.

b.



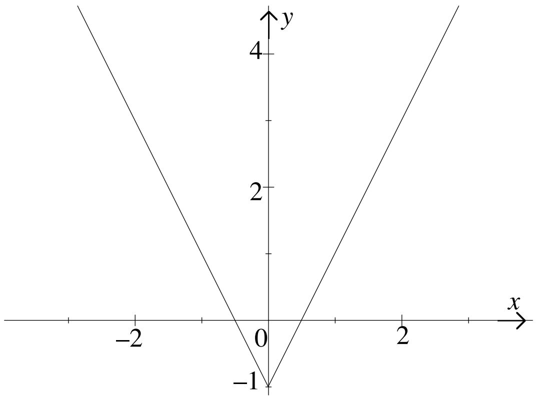
The graph of $y = |2x|$. Its domain is all real x and range all real $y \geq 0$.

c.



The graph of $y = \frac{1}{x-4}$. The domain is all real $x \neq 4$ and the range is all real $y \neq 0$.

d.

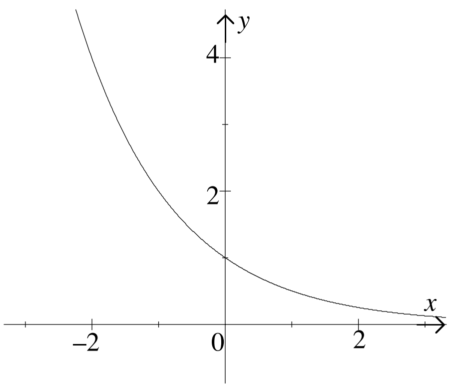


The graph of $y = |2x| - 1$. The domain is all real x, and the range is all real $y \geq -1$.

4. a. The perpendicular distance d from $(0, 0)$ to $x + y + k = 0$ is $d = |\frac{k}{\sqrt{2}}|$.

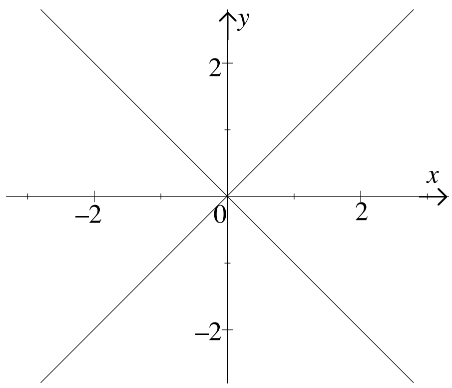
b. For the line $x+y+k = 0$ to cut the circle in two distinct points $d < 2$. ie $|k| < 2\sqrt{2}$ or $-2\sqrt{2} < k < 2\sqrt{2}$.

5. a.



The graph of $y = (\frac{1}{2})^x$.

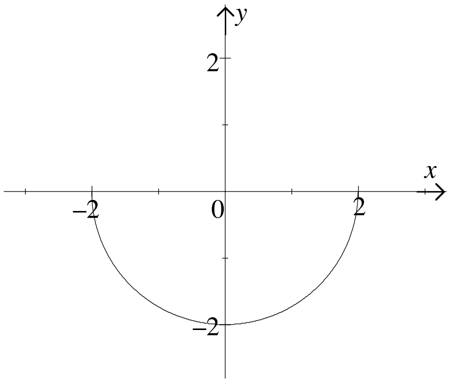
b.



The graph of $y^2 = x^2$.

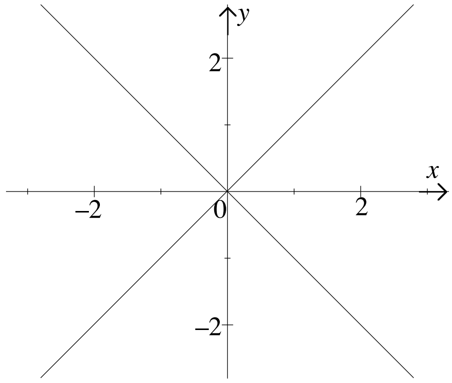
6. $y^2 = x^3$ is not a function.

7. a.



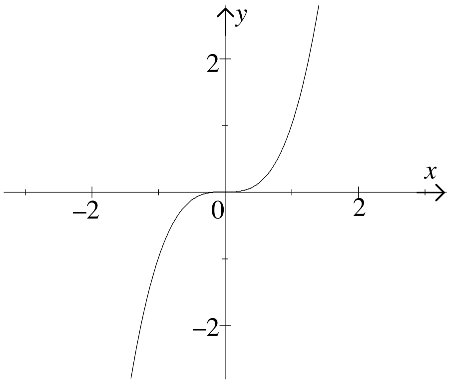
The graph of $y = -\sqrt{4 - x^2}$. This is a function with the domain: all real x such that $-2 \le x \le 2$ and range: all real y such that $-2 \le y \le 0$.

b.



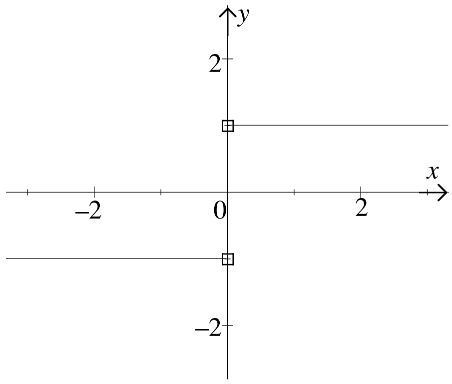
The graph of $|x| - |y| = 0$. This is not the graph of a function.

c.



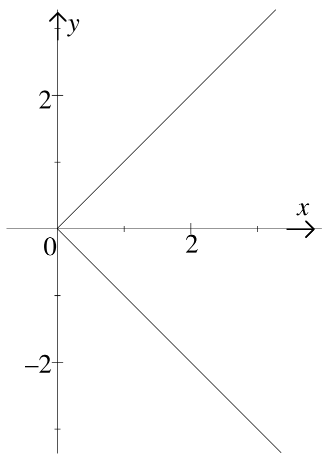
The graph of $y = x^3$. This is a function with the domain: all real x and range: all real y.

d.



The graph of $y = \frac{x}{|x|}$. This is the graph of a function which is not defined at $x = 0$. Its domain is all real $x \neq 0$, and range is $y = \pm 1$.

e.



The graph of $|y| = x$. This is not the graph of a function.

8.

$$\begin{aligned} A(\frac{1}{p}) &= (\frac{1}{p})^2 + 2 + \frac{1}{(\frac{1}{p})^2} \\ &= \frac{1}{p^2} + 2 + \frac{1}{\frac{1}{p^2}} \\ &= \frac{1}{p^2} + 2 + p^2 \\ &= A(p) \end{aligned}$$

9. **a.** The values of x in the interval $0 < x < 4$ are not in the domain of the function.

b. $x = 1$ and $x = -1$ are not in the domain of the function.

10. **a.** $\phi(3) + \phi(4) + \phi(5) = \log(2.5)$

b. $\phi(3) + \phi(4) + \phi(5) + \cdots + \phi(n) = \log(\frac{n}{2})$

11. **a.** $y = 3$ when $z = 3$.

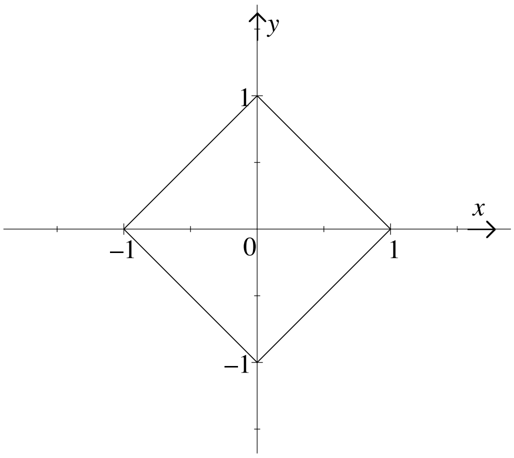
b. i $L(M(x)) = 2(x^2 - x) + 1$

ii $M(L(x)) = 4x^2 + 2x$

12. **a.** $a = 2$, $b = 2$ so the equations is $y = 2x^2 - 2$.

b. $a = 5$, $b = 1$ so the equation is $y = \frac{5}{x^2+1}$.

13. **b.**



The graph of $|x| + |y| = 1$.

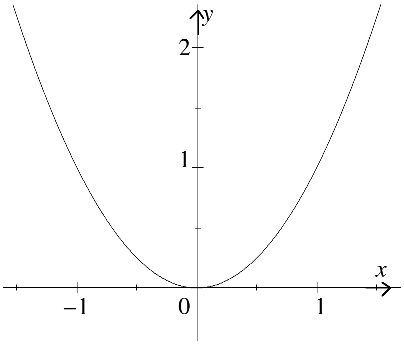
14. $S(n-1) = \frac{n-1}{2n-1}$

Hence

$$\begin{aligned} S(n) - S(n-1) &= \frac{n}{2n+1} - \frac{n-1}{2n-1} \\ &= \frac{n(2n-1) - (2n+1)(n-1)}{(2n-1)(2n+1)} \\ &= \frac{2n^2 - n - (2n^2 - n - 1)}{(2n-1)(2n+1)} \\ &= \frac{1}{(2n-1)(2n+1)} \end{aligned}$$

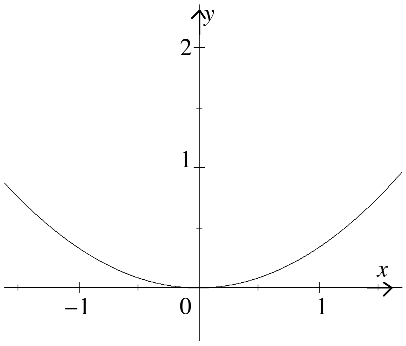
2.8 Solutions

1. a.



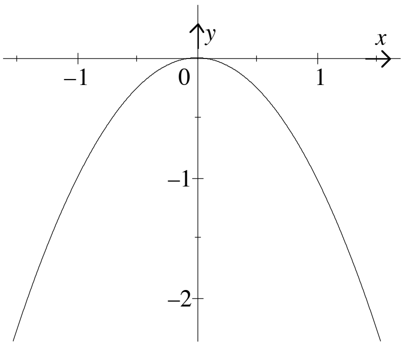
The graph of $y = x^2$.

b.



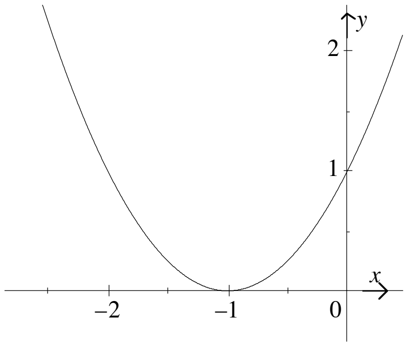
The graph of $y = \frac{x^2}{3}$.

c.



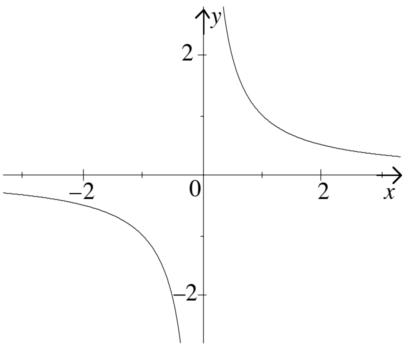
The graph of $y = -x^2$.

d.



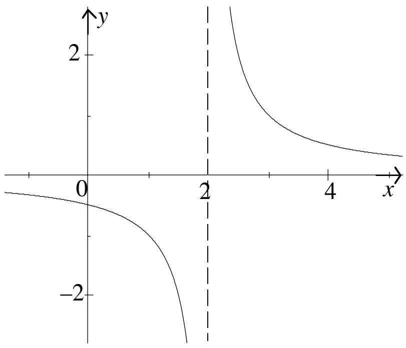
The graph of $y = (x + 1)^2$.

2. a.



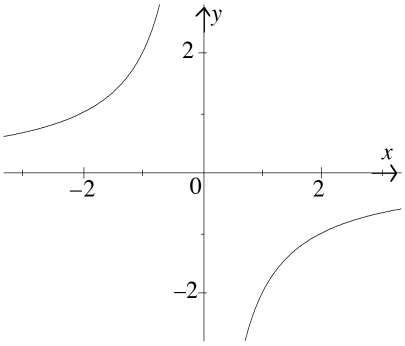
The graph of $y = \frac{1}{x}$.

b.



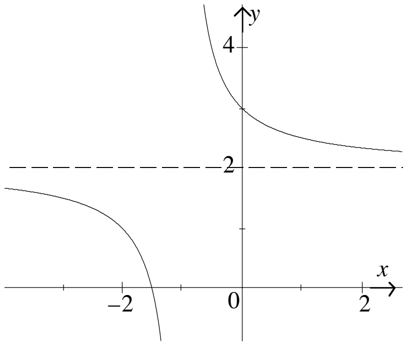
The graph of $y = \frac{1}{x-2}$.

c.



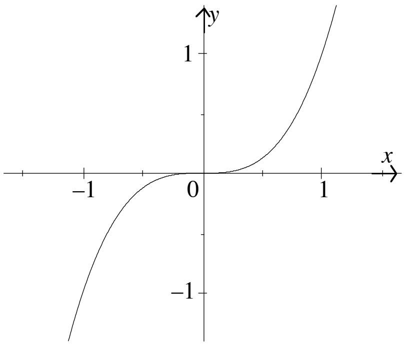
The graph of $y = \frac{-2}{x}$.

d.



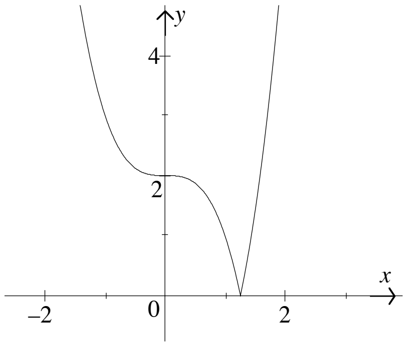
The graph of $y = \frac{1}{x+1} + 2$.

3. a.



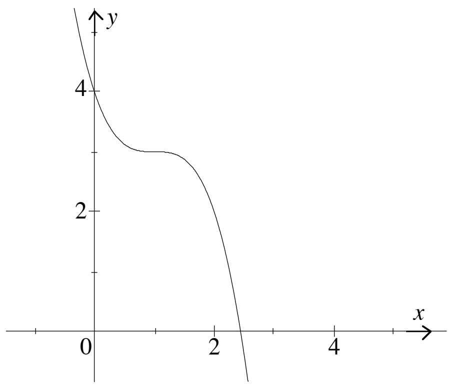
The graph of $y = x^3$.

b.



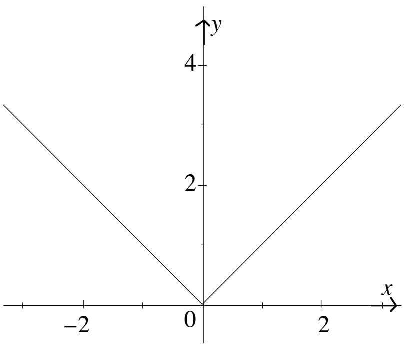
The graph of $y = |x^3 - 2|$.

c.



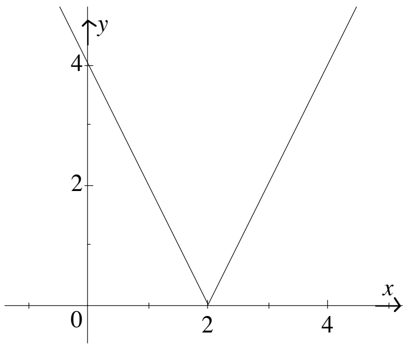
The graph of $y = 3 - (x - 1)^3$.

4. a.



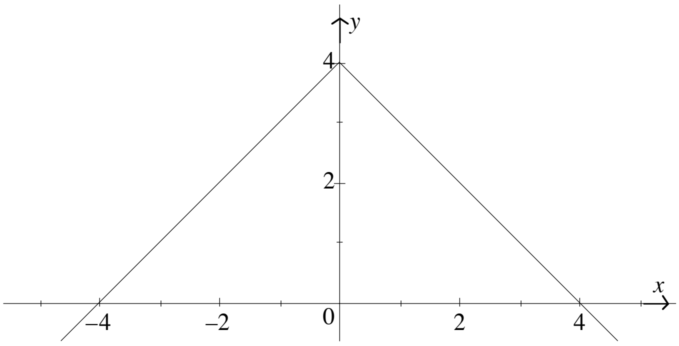
The graph of $y = |x|$.

b.



The graph of $y = 2|x - 2|$.

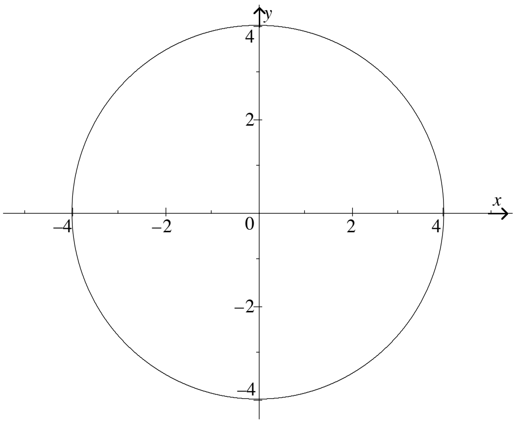
c.



The graph of $y = 4 - |x|$.

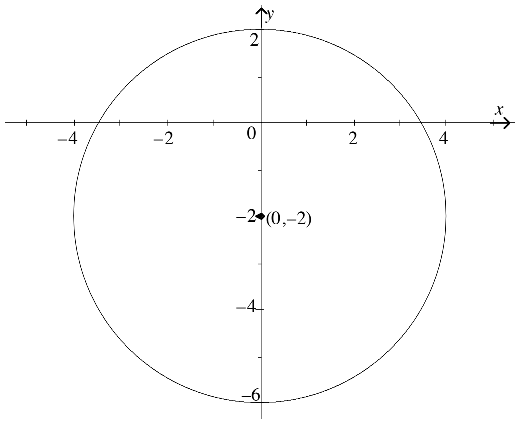
5.

a.



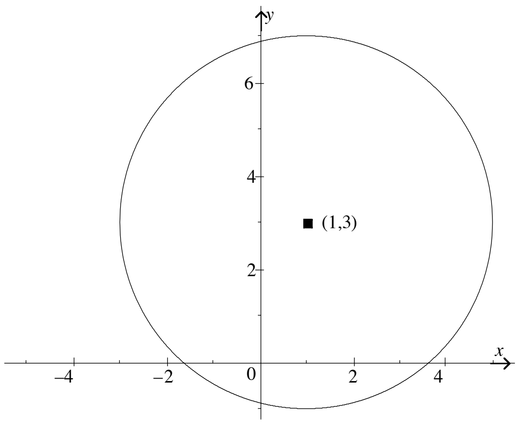
The graph of $x^2 + y^2 = 16$.

b.



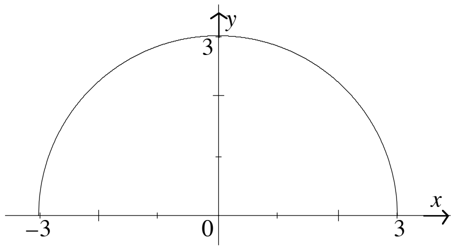
The graph of $x^2 + (y + 2)^2 = 16$.

c.



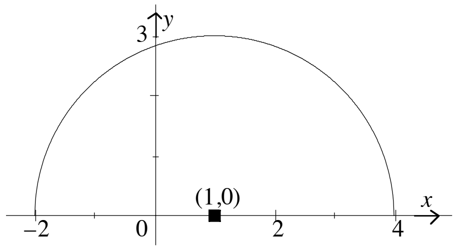
The graph of $(x - 1)^2 + (y - 3)^2 = 16$.

6. a.



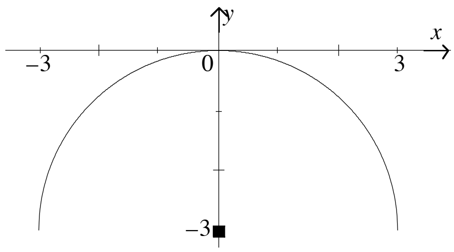
The graph of $y = \sqrt{9 - x^2}$.

b.



The graph of $y = \sqrt{9 - (x-1)^2}$.

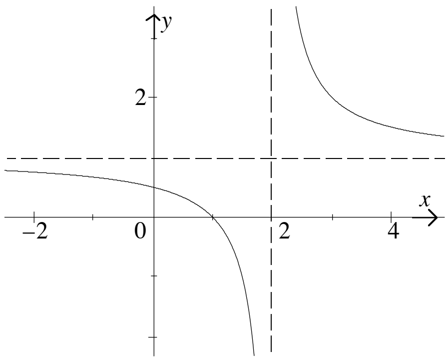
c.



The graph of $y = \sqrt{9 - x^2} - 3$.

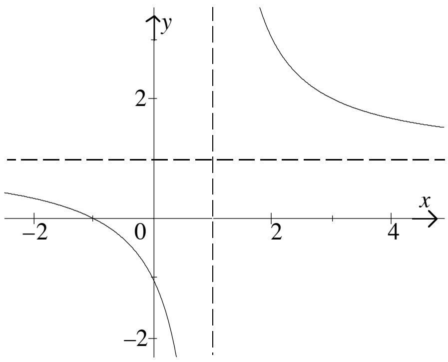
7.

$$\frac{1}{x-2} + 1 = \frac{1 + (x-2)}{x-2} = \frac{x-1}{x-2}$$



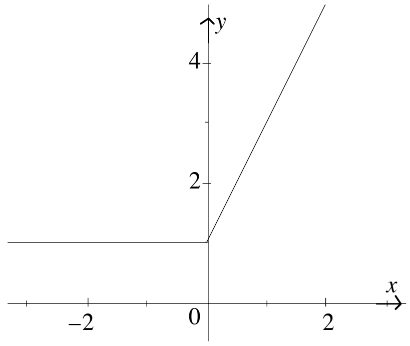
The graph of $y = \frac{x-1}{x-2}$.

8.



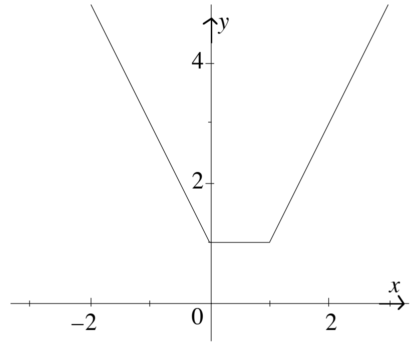
The graph of $y = \frac{x+1}{x-1}$.

9. a.



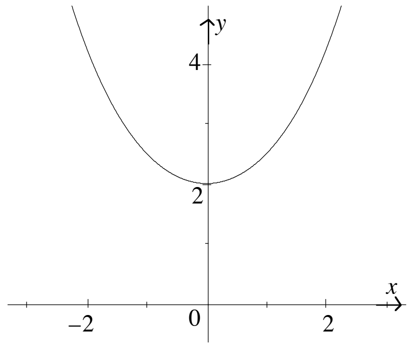
The graph of $y = |x| + x + 1$ for $-2 \leq x \leq 2$.

b.



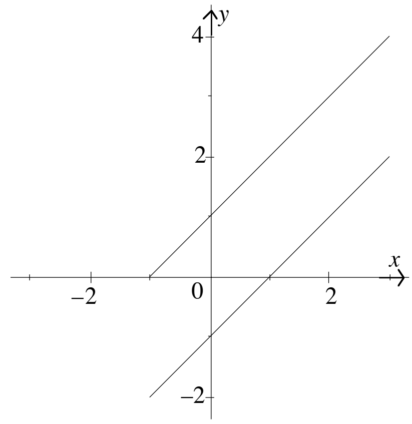
The graph of $y = |x| + |x - 1|$ for $-2 \leq x \leq 3$.

c.



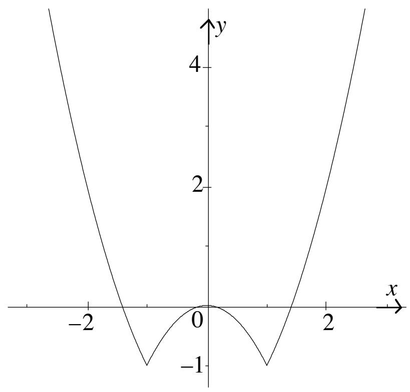
The graph of $y = 2^x + 2^{-x}$ for $-2 \leq x \leq 2$.

d.



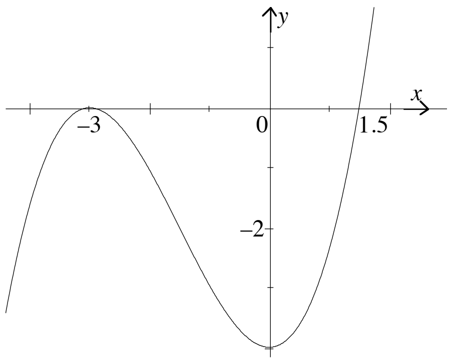
The graph of $|x - y| = 1$ for $-1 \leq x \leq 3$.

10.



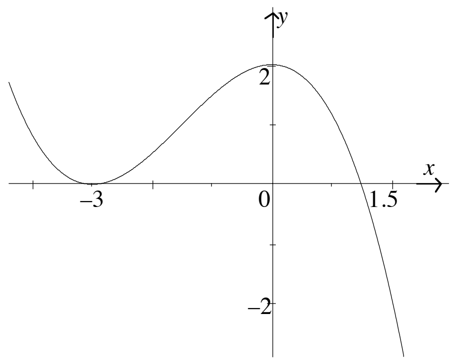
The graph of $f(x) = |x^2 - 1| - 1$.

11. a.



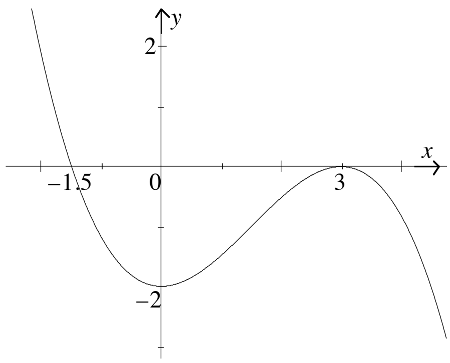
The graph of $y = 2f(x)$.

b.



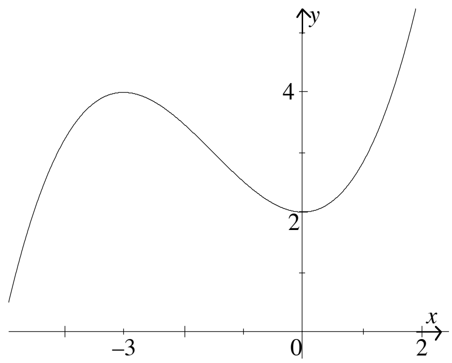
The graph of $y = -f(x)$.

c.



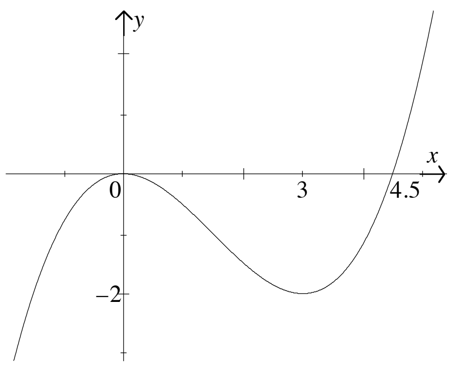
The graph of $y = f(-x)$.

d.



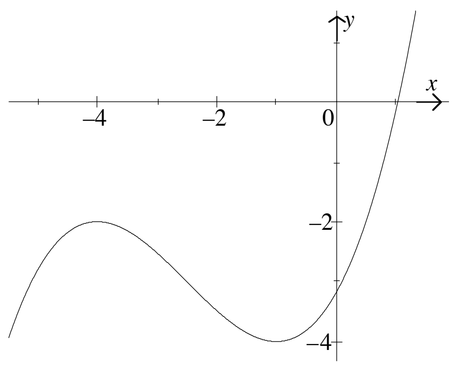
The graph of $y = f(x) + 4$.

e.



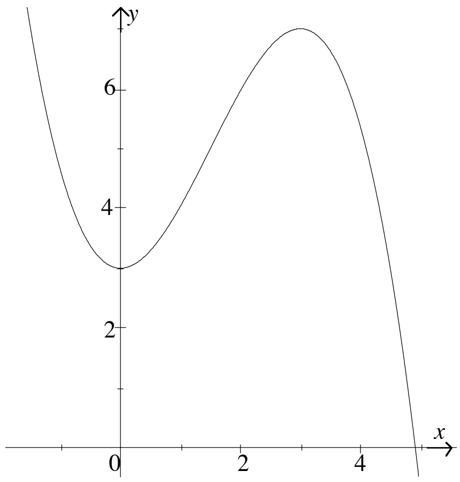
The graph of $y = f(x - 3)$.

f.



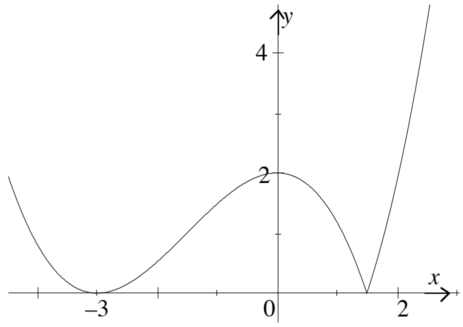
The graph of $y = f(x + 1) - 2$.

g.



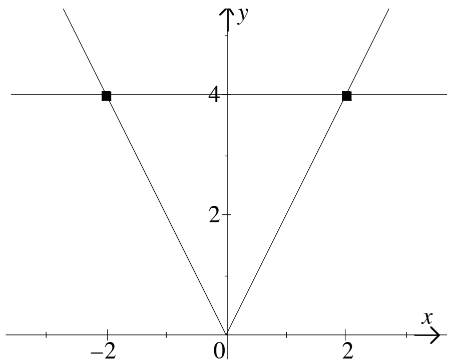
The graph of $y = 3 - 2f(x-3)$.

h.



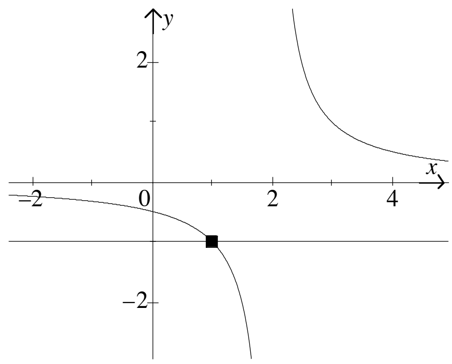
The graph of $y = |f(x)|$.

12. a.



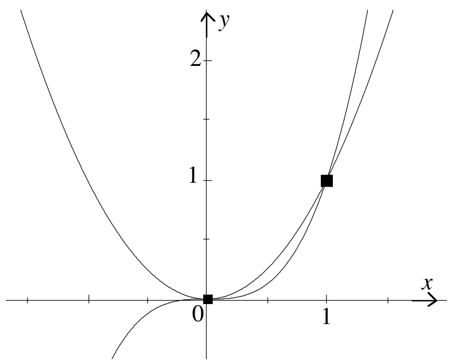
$x = -2$ and $x = 2$ are solutions of the equation $|2x| = 4$.

b.



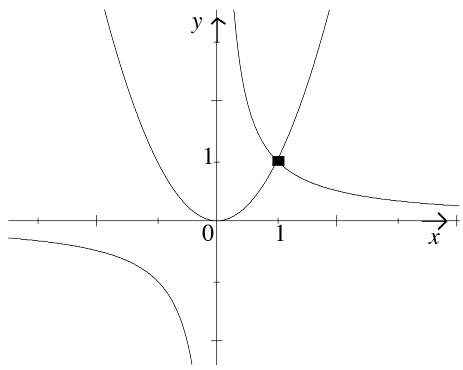
$x = 1$ is a solution of $\frac{1}{x-2} = -1$.

c.



$x = 0$ and $x = 1$ are solutions of the equation $x^3 = x^2$.

d.



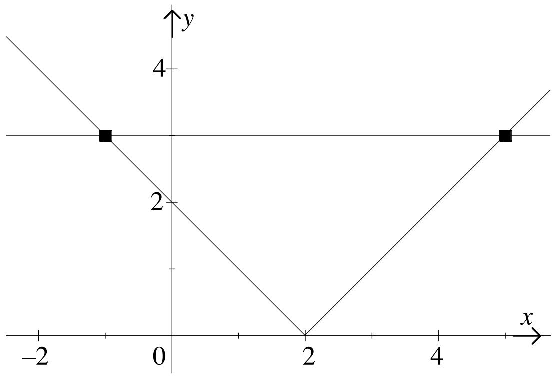
$x = 1$ is a solution of $x^2 = \frac{1}{x}$.

13.

a. For $x \geq 2$, $|x-2| = x-2 = 3$. Therefore $x = 5$ is a solution of the inequality. (Note that $x = 5$ is indeed ≥ 2.)

For $x < 2$, $|x-2| = -(x-2) = -x+2 = 3$. Therefore $x = -1$ is a solution. (Note that $x = -1$ is < 2.)

b.

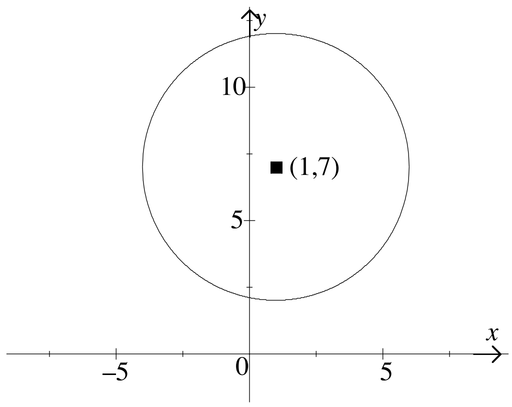


The points of intersection are $(-1, 3)$ and $(5, 3)$.

Therefore the solutions of $|x - 2| = 3$ are $x = -1$ and $x = 5$.

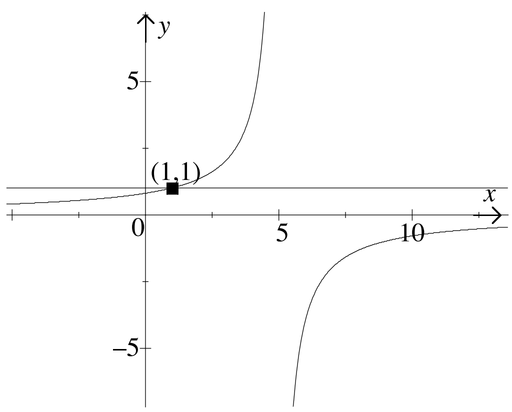
14. The parabolas intersect at $(2, 1)$.

15.



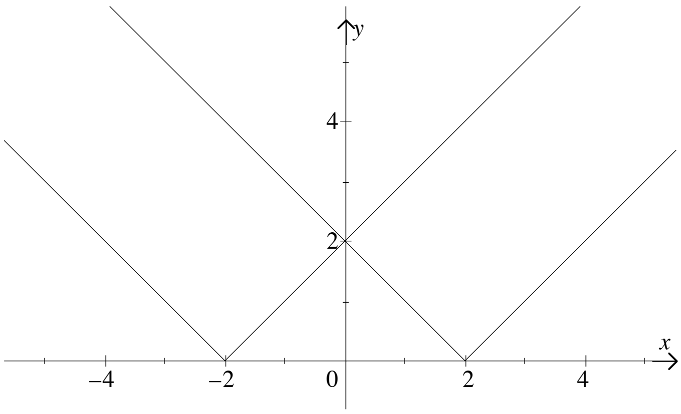
$y = k$ intersects the circle at two distinct points when $2 < k < 12$.

16.



The point of intersection is $(1, 1)$. Therefore the solution of $\frac{4}{5-x} = 1$ is $x = 1$.

17.

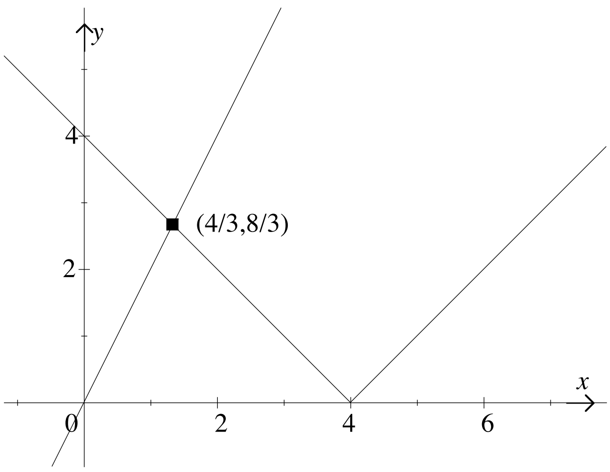


The point of intersection is $(0, 2)$. Therefore the solution of $|x-2| = |x+2|$ is $x = 0$.

18. $n = -1$ or $n = 2$.

19. a. For $x \geq 4$, $|x-4| = x-4 = 2x$ when $x = -4$, but this does not satisfy the condition of $x \geq 4$ so is not a solution.
For $x < 4$, $|x-4| = -x+4 = 2x$ when $x = \frac{4}{3}$. $x = \frac{4}{3}$ is < 4 so is a solution.
Therefore, $x = \frac{4}{3}$ is a solution of $|x-4| = 2x$.

b.



The graph of $y = |x-4|$ and $y = 2x$ intersect at the point $(\frac{4}{3}, \frac{8}{3})$. So the solution of $|x-4| = 2x$ is $x = \frac{4}{3}$.

2.11 Solutions

1. a. The domain is all real x, and the range is all real $y \geq -2$.

b. i $-2 < x < 0$ or $x > 2$

ii $x < -2$ or $0 < x < 2$

c. i $k < -2$

ii There is no value of k for which $f(x) = k$ has exactly one solution.

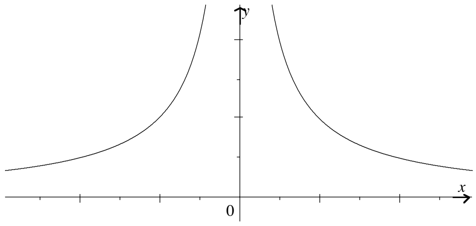
iii $k = 2$ or $k > 0$

iv $k = 0$

v $-2 < k < 0$

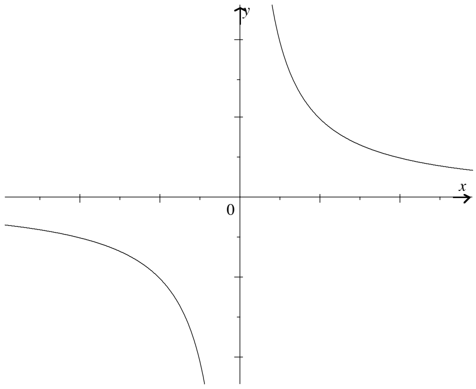
d. $y = f(x)$ is even

2. a.



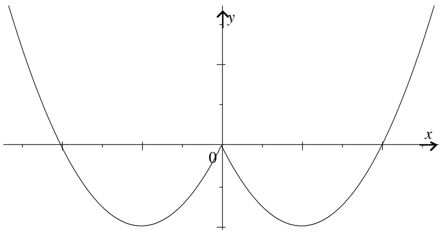
$y = f(x)$ is even.

b.



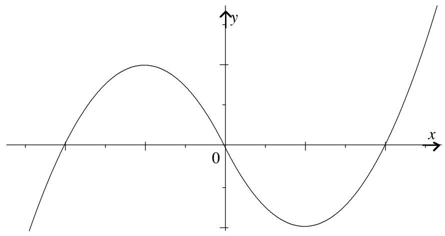
$y = f(x)$ is odd.

a.



$y = g(x)$ is even.

b.



$y = g(x)$ is odd.

3. a.	even	**b.**	even	**c.**	neither	**d.**	odd	**e.**	odd
f.	even	**g.**	even	**h.**	neither	**i.**	even	**j.**	even

4. a.

$$\begin{aligned} h(-x) &= f(-x) \cdot g(-x) \\ &= f(x) \cdot -g(x) \\ &= -f(x) \cdot g(x) \\ &= -h(x) \end{aligned}$$

Therefore h is odd.

b.

$$\begin{aligned} h(-x) &= (g(-x))^2 \\ &= (-(g(x))^2 \\ &= (g(x))^2 \\ &= h(x) \end{aligned}$$

Therefore h is even.

c.

$$\begin{aligned} h(-x) &= \frac{f(-x)}{g(-x)} \\ &= \frac{f(x)}{-g(x)} \\ &= -\frac{f(x)}{g(x)} \\ &= -h(x) \end{aligned}$$

Therefore h is odd.

d.

$$\begin{aligned} h(-x) &= f(-x)\cdot(g(-x))^2 \\ &= f(x)\cdot(-g(x))^2 \\ &= f(x)\cdot(g(x))^2 \\ &= h(x) \end{aligned}$$

Therefore h is even.

5. If f is defined at $x = 0$

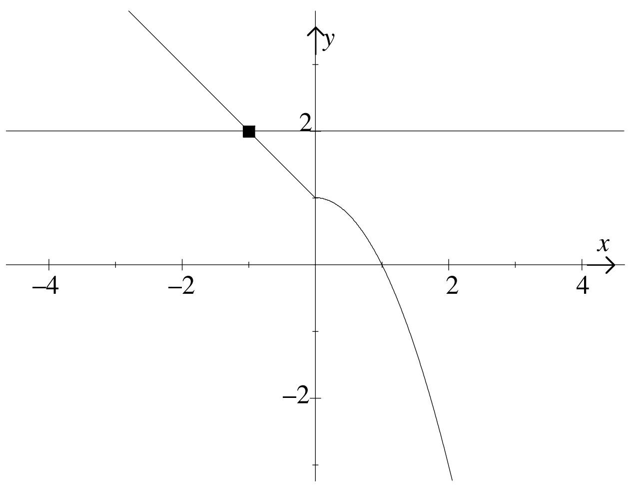
$$\begin{aligned} f(0) &= f(-0) && \text{(since } 0 = -0) \\ &= -f(0) && \text{(since } f \text{ is odd)} \\ 2f(0) &= 0 && \text{(adding } f(0) \text{ to both sides)} \\ \text{Therefore } f(0) &= 0. \end{aligned}$$

3.2 Solutions

1. a. $2f(-1) + f(2) = 2(1 - (-1)) + (1 - (2)^2) = 4 + (-3) = 1.$

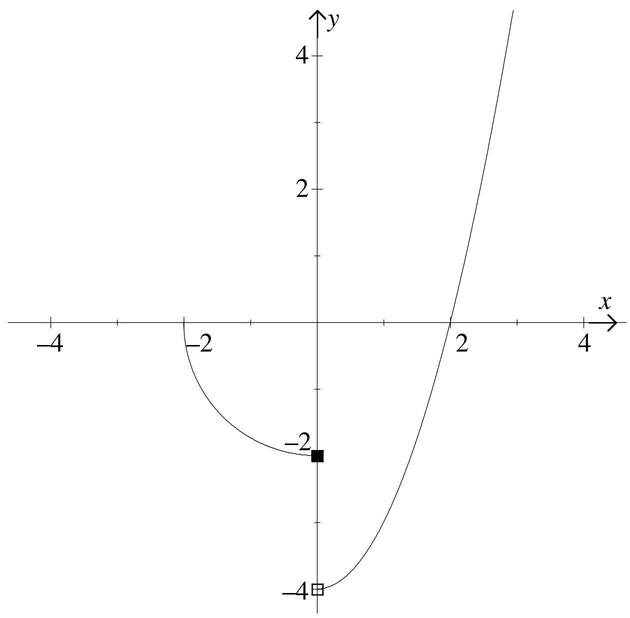
b. $f(a^2) = 1 - (a^2)^2 = 1 - a^4$ since $a^2 \geq 0.$

2. You can see from the graph below that there is one solution to $f(x) = 2$, and that this solution is at $x = -1$.



3. $$g(x) = \begin{cases} \frac{1}{x+1} & \text{for } x < 1 \\ \sqrt{1-x^2} & \text{for } -1 \le x \le 1 \\ -1 & \text{for } x > 1 \end{cases}$$

4. **a.** The domain of f is all real $x \ge -2$.



b. The range of f is all real $y > -4$.

c. i $f(x) = 0$ when $x = -2$ or $x = 2$.

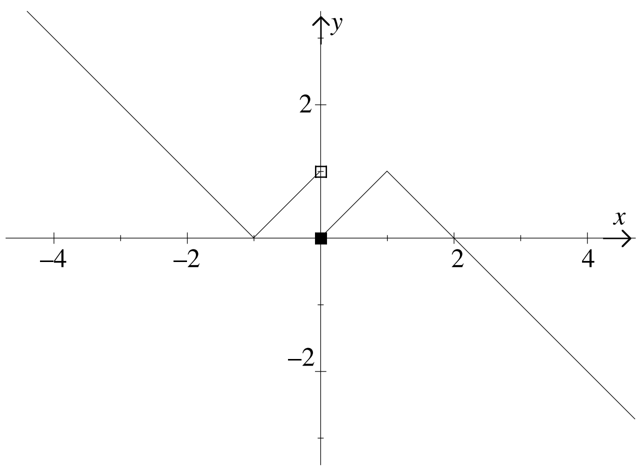
ii $f(x) = -3$ when $x = 1$.

d. i $f(x) = k$ has no solutions when $k \le -4$.

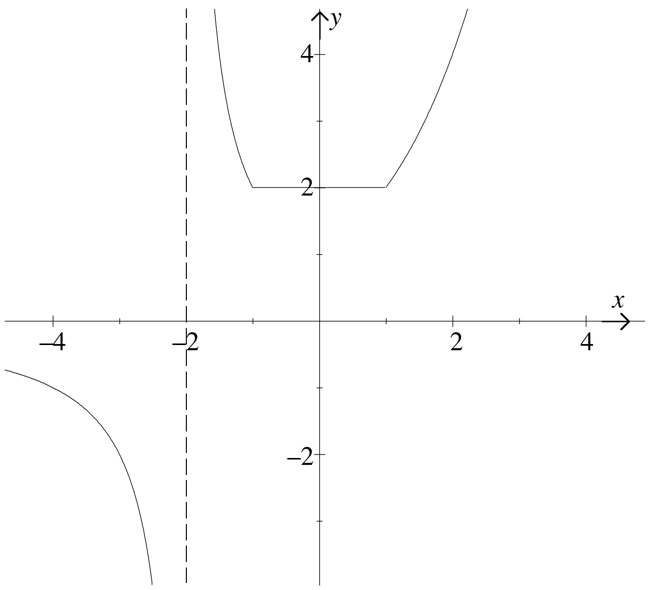
ii $f(x) = k$ has 1 solution when $-4 < k < -2$ or $k > 0$.

iii $f(x) = k$ has 2 solutions when $-2 \le k \le 0$.

5. Note that $f(0) = 0$.



6. The domain of g is all real x, $x \ne -2$.



The range of g is all real $y < 0$ or $y \geq 2$.

7. Note that there may be more than one correct solution.

a. Defining f as

$$f(x) = \begin{cases} x + 6 & \text{for } x \leq -3 \\ -x & \text{for } -3 < x < 0 \\ x & \text{for } 0 \leq x \leq 3 \\ -x + 6 & \text{for } x > 3 \end{cases}$$

gives a function describing the McMaths burgers' logo using 4 pieces.

b. Defining f as

$$f(x) = \begin{cases} x + 6 & \text{for } x \leq -3 \\ |x| & \text{for } -3 < x < 3 \\ -x + 6 & \text{for } x \geq 3 \end{cases}$$

gives a function describing the McMaths burgers' logo using 3 pieces.

c. Defining f as

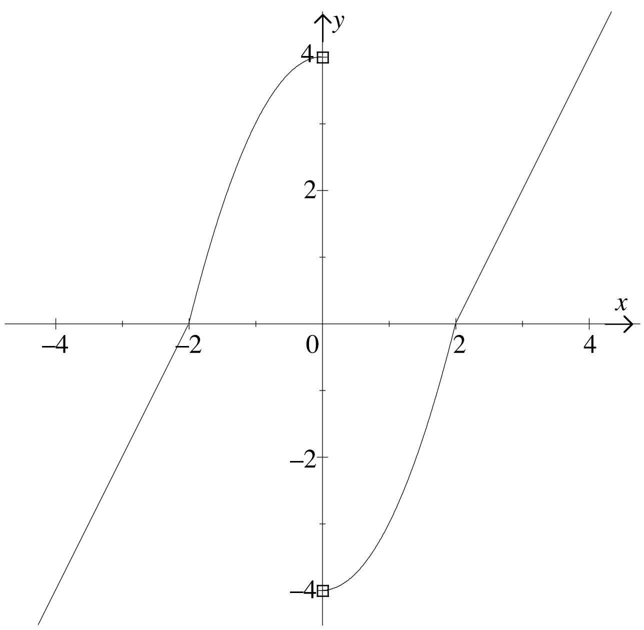
$$f(x) = \begin{cases} 3 - |x + 3| & \text{for } x \leq 0 \\ 3 - |x - 3| & \text{for } x > 0 \end{cases}$$

gives a function describing the McMaths burgers' logo using 2 pieces.

8. **a.** Here $a = 1$, $b = -4$, $c = 2$ and $d = -4$. So,

$$f(x) = \begin{cases} x^2 - 4 & \text{for } 0 < x \leq 2 \\ 2x - 4 & \text{for } x > 2 \end{cases}$$

b. Defining f to be an odd function for all real x, $x \neq 0$, we get



c. We can define f as follows

$$f(x) = \begin{cases} 2x+4 & \text{for } x < -2 \\ 4-x^2 & \text{for } -2 \le x < 0 \\ x^2-4 & \text{for } 0 < x \le 2 \\ 2x-4 & \text{for } x > 2 \end{cases}$$

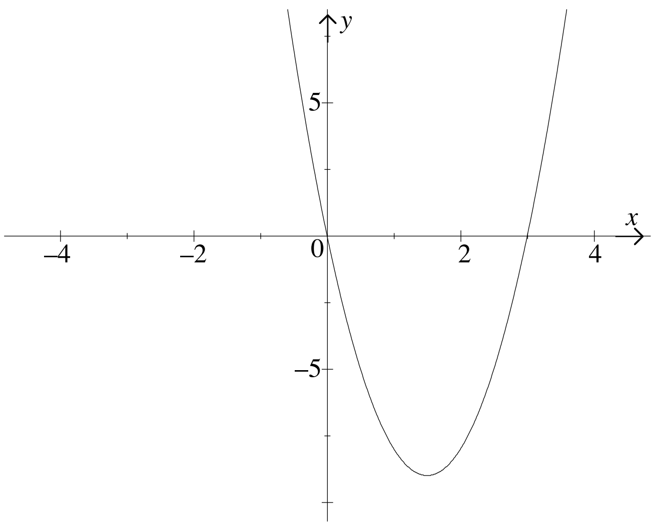
3.4 Solutions

1. **a.** $0 \le x \le 4$

b. $-3 < p \le 1$

c. $x < -4$ or $-3 < x < 3$ or $x > 4$

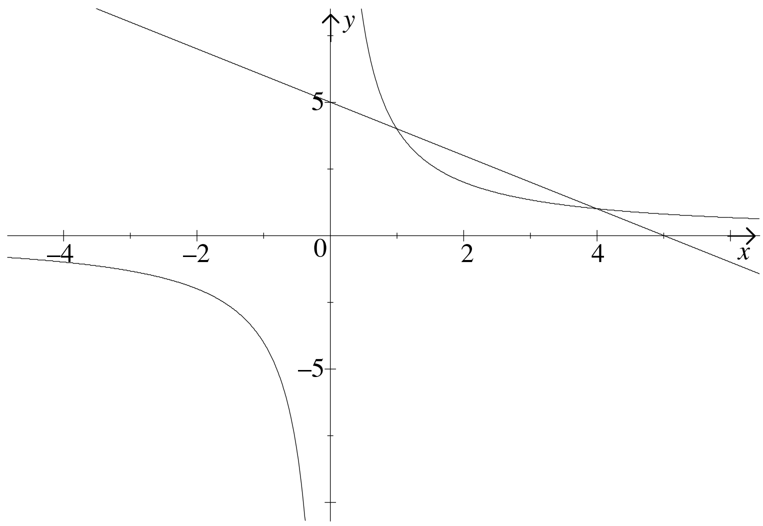
2. **a.** The graph of $y = 4x(x-3)$ is given below



b. From the graph we see that $4x(x-3) \le 0$ when $0 \le x \le 3$.

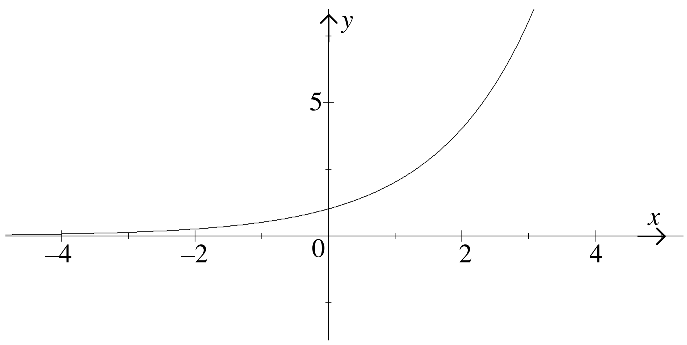
3. a. The graphs $y = 5 - x$ and $y = \frac{4}{x}$ intersect at the points $(1, 4)$ and $(4, 1)$.

b. The graphs of $y = 5 - x$ and $y = \frac{4}{x}$



c. The inequality is satisfied for $x < 0$ or $1 < x < 4$.

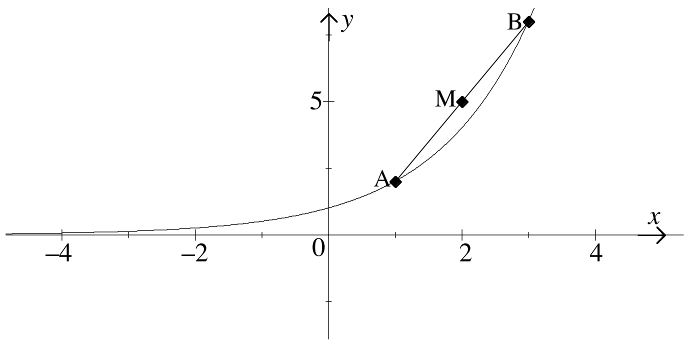
4. a. The graph of $y = 2^x$.



b. $2^x < \frac{1}{2}$ when $x < -1$.

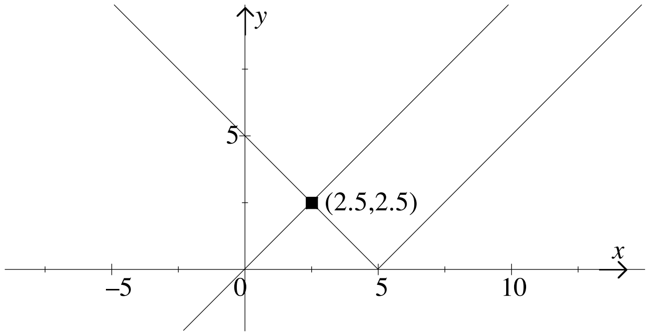
c. The midpoint M of the segment AB has coordinates $(\frac{a+b}{2}, \frac{2^a+2^b}{2})$.
Since the function $y = 2^x$ is concave up, the y-coordinate of M is greater than $f(\frac{a+b}{2})$. So,

$$\frac{2^a + 2^b}{2} > 2^{\frac{a+b}{2}}$$



5.

a.



b. $|x-5| > x$ for all $x < 2.5$.

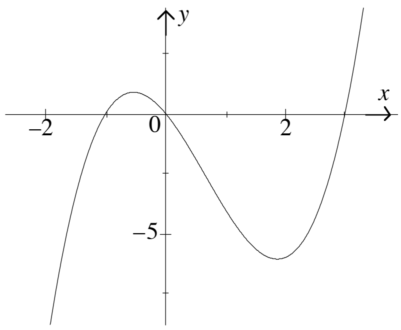
c. i $\quad mx = |x-5|$ has exactly two solutions when $0 < m < 1$.

ii $\quad mx = |x-5|$ has no solutions when $-1 < m < 0$.

6. $-1 \leq x \leq 3$

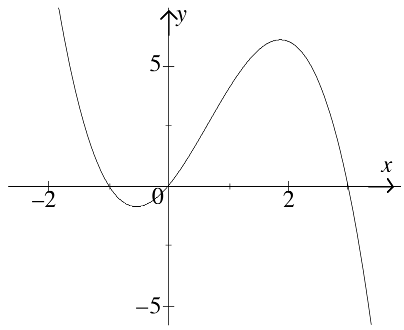
4.3 Solutions

1. a.



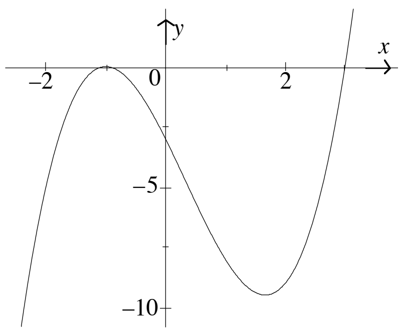
The graph of $P(x) = x(x+1)(x-3)$.

b.



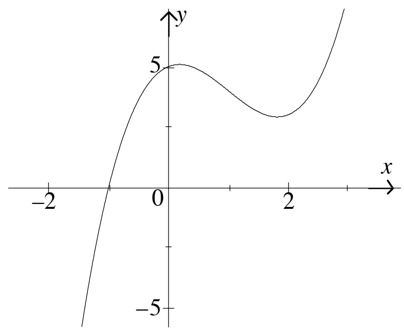
The graph of $P(x) = x(x+1)(3-x)$.

c.



The graph of $P(x) = (x+1)^2(x-3)$.

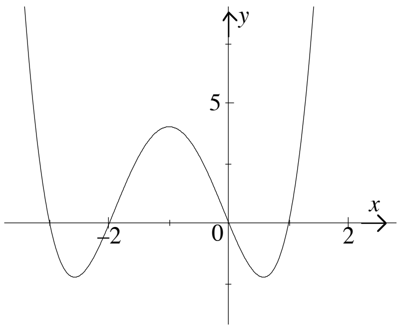
d.



The graph of
$P(x) = (x+1)(x^2-4x+5)$.

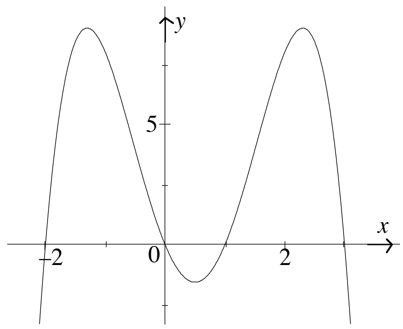
2. a. iv. **b.** v. **c.** i. **d.** iii. **e.** ii. **f.** vi.

3. a.



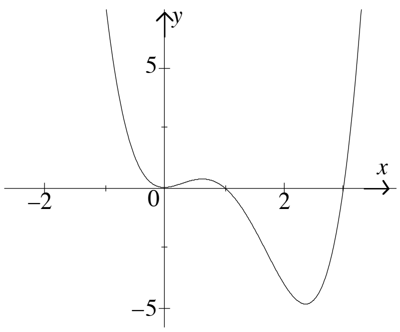
The graph of
$P(x) = x(x-1)(x+2)(x+3)$.

b.



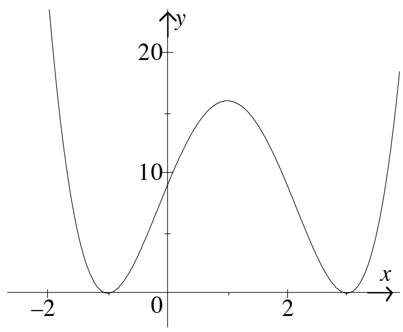
The graph of
$P(x) = x(x-1)(x+2)(3-x)$.

c.



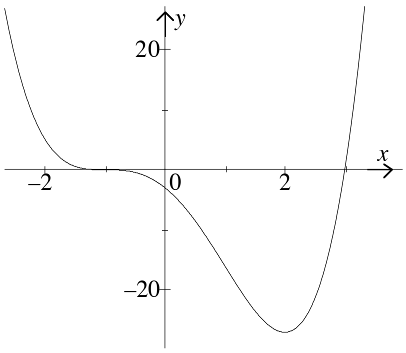
The graph of $P(x) = x^2(x-1)(x-3)$.

d.



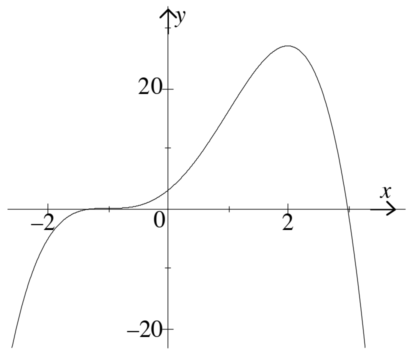
The graph of $P(x) = (x+1)^2(x-3)^2$.

e.



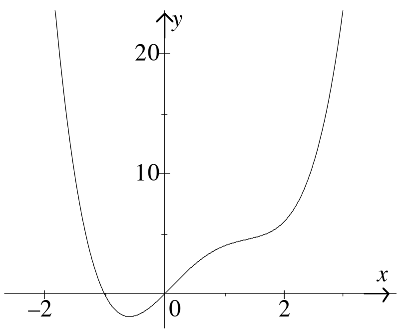
The graph of $P(x) = (x+1)^3(x-3)$.

f.



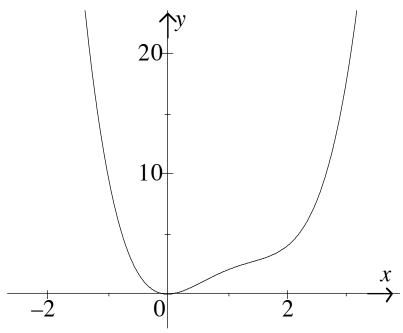
The graph of $P(x) = (x+1)^3(3-x)$.

g.



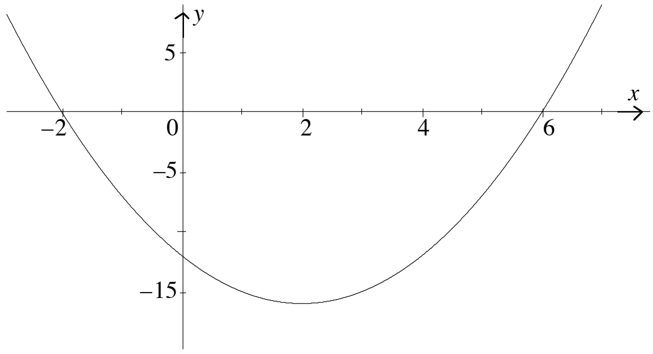
The graph of
$P(x) = x(x+1)(x^2 - 4x + 5)$.

h.



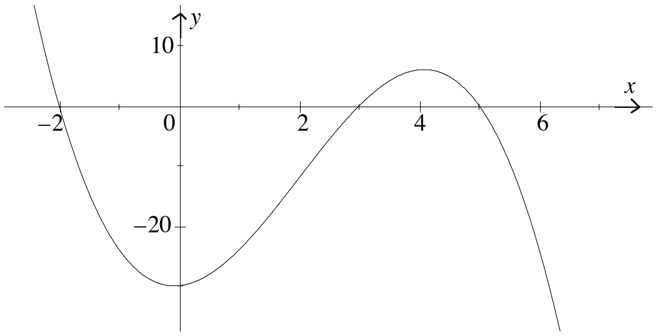
The graph of $P(x) = x^2(x^2 - 4x + 5)$.

4. a.



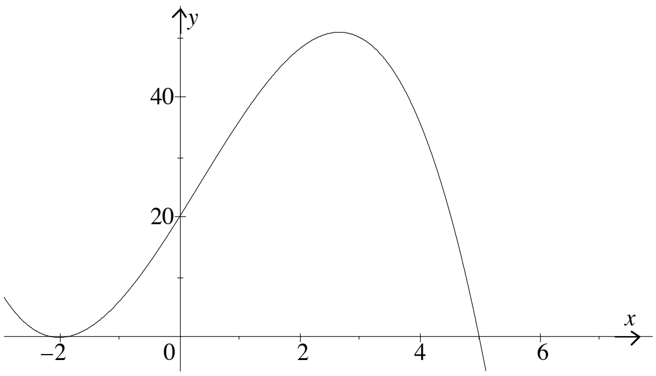
$x^2 - 4x - 12 < 0$ when $-2 < x < 6$.

b.



$(x + 2)(x - 3)(5 - x) > 0$ when $x < -2$ or $3 < x < 5$.

c.



$(x + 2)^2(5 - x) > 0$ when $x < 5$.

5. $x^2 - 4x - 12 + k \geq 0$ for all real x when $k = 16$.

6. a. $P(x) = x(x - 4)$

b. $P(x) = -x(x - 4)$

c. $P(x) = x^2(x - 4)$

d. $P(x) = \frac{x^3(x-4)}{3}$

e. $P(x) = -x(x - 4)^2$

f. $P(x) = \frac{(x+4)(x-4)^2}{8}$

7. a. The roots of $f(x) = 0$ are $x = -2$, $x = 0$ and $x = 2$.

b. $x = 2$ is the repeated root.

c. The equation $f(x) = k$ has exactly 3 solutions when $k = 0$ or $k = 3.23$.

d. $f(x) < 0$ when $-2 < x < 0$.

e. The least possible degree of the polynomial $f(x)$ is 4.

f. Since $f(0) = 0$, the constant in the polynomial is 0.

g. $f(x) + k \geq 0$ for all real x when $k \geq 9.91$.

4.5 Solutions

1. a. Since $A(x) = (x - a)(x - b)$ is a polynomial of degree 2, the remainder $R(x)$ must be a polynomial of degree < 2. So, $R(x)$ is a polynomial of degree ≤ 1. That is, $R(x) = mx + c$ where m and c are constants. Note that if $m = 0$ the remainder is a constant.

b. Let $P(x) = (x^2 - 5x + 6)Q(x) + (mx + c) = (x - 2)(x - 3)Q(x) + (mx + c)$.

Then

$$\begin{aligned} P(2) &= (0)(-1)Q(2) + (2m + c) \\ &= 2m + c \\ &= 4 \end{aligned}$$

and

$$\begin{aligned} P(3) &= (1)(0)Q(3) + (3m + c) \\ &= 3m + c \\ &= 9 \end{aligned}$$

Solving simultaneously we get that $m = 5$ and $c = -6$. So, the remainder is $R(x) = 5x - 6$.

c. Let $P(x) = (x - a)(x - b)Q(x) + (mx + c)$.

Then

$$\begin{aligned} P(a) &= (0)(a - b)Q(a) + (ma + c) \\ &= am + c \\ &= a^2 \end{aligned}$$

and

$$\begin{aligned} P(b) &= (b - a)(0)Q(b) + (mb + c) \\ &= bm + c \\ &= b^2 \end{aligned}$$

Solving simultaneously we get that $m = a + b$ and $c = -ab$ provided $a \neq b$.

So, $R(x) = (a + b)x - ab$.

2. a.

$$2x^4 + 13x^3 + 18x^2 + x - 4 = (x^2 + 5x + 2)(2x^2 + 3x - 1) - 2$$

b. Let α be a common zero of $f(x)$ and $g(x)$. That is, $f(\alpha) = 0$ and $g(\alpha) = 0$.

Then since $f(x) = g(x)q(x) + r(x)$ we have

$$\begin{aligned} f(\alpha) &= g(\alpha)q(\alpha) + r(\alpha) \\ &= (0)q(\alpha) + r(\alpha) \qquad \text{since } g(\alpha) = 0 \\ &= r(\alpha) \\ &= 0 \qquad \text{since } f(\alpha) = 0 \end{aligned}$$

But, from part **b.** $r(x) = -2$ for all values of x, so we have a contradiction.

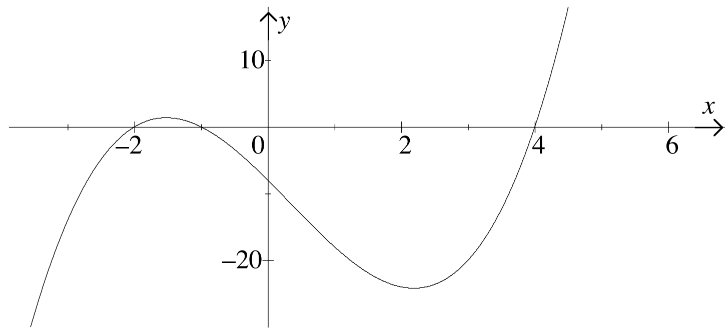
Therefore, $f(x)$ and $g(x)$ do not have a common zero.

This is an example of a proof by contradiction.

3. **a.** i $P(x) = x^3 - x^2 - 10x - 8 = (x+1)(x+2)(x-4)$

ii $x = -1$, $x = -2$ and $x = 4$ are solutions of $P(x) = 0$.

iii

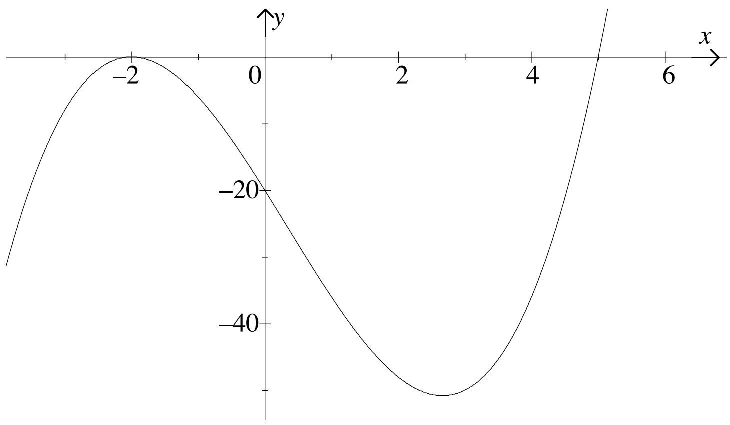


The graph of $P(x) = x^3 - x^2 - 10x - 8$.

b. i $P(x) = x^3 - x^2 - 16x - 20 = (x+2)^2(x-5)$.

ii $x = -2$ and $x = 5$ are solutions of $P(x) = 0$. $x = -2$ is a double root.

iii

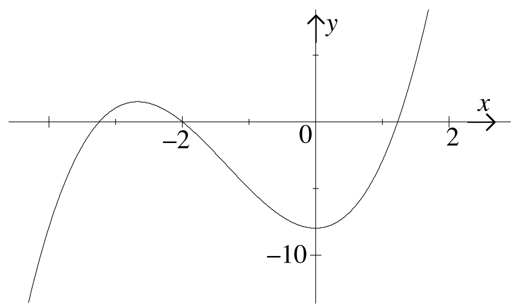


The graph of $P(x) = x^3 - x^2 - 16x - 20$.

c. i $P(x) = x^3+4x^2-8 = (x+2)(x^2+2x-4) = (x+2)(x-(-1+\sqrt{5}))(x-(-1-\sqrt{5}))$

ii $x = -2$, $x = -1+\sqrt{5}$ and $x = -1-\sqrt{5}$ are solutions of $P(x) = 0$.

iii



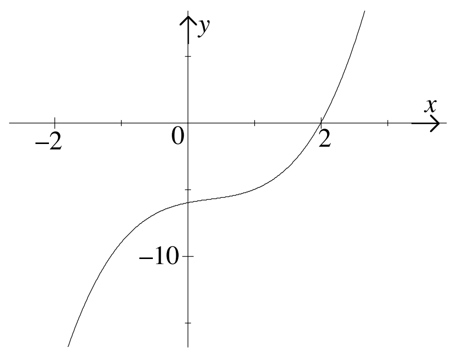
The graph of $P(x) = x^3 + 4x^2 - 8$.

The zeros are $x = -2$, $x = -1 + \sqrt{5}$ and $x = -1 - \sqrt{5}$.

d. i $P(x) = x^3 - x^2 + x - 6 = (x - 2)(x^2 + x + 3)$. $x^2 + x + 3 = 0$ has no real solutions.

ii $x = 2$ is the only real solution of $P(x) = 0$.

iii



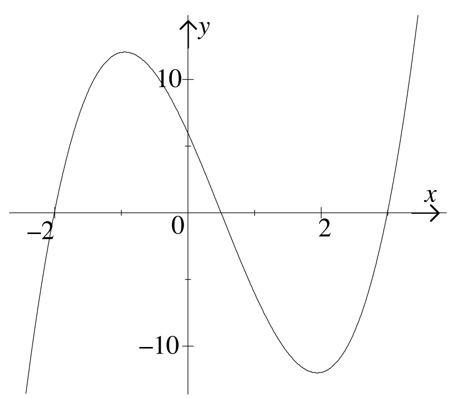
The graph of $P(x) = x^3 - x^2 + x - 6$.

There is only one real zero at $x = 2$.

e. i $P(x) = 2x^3 - 3x^2 - 11 + 6 = (x + 2)(x - 3)(2x - 1)$.

ii $x = -2$, $x = \frac{1}{2}$ and $x = 3$ are solutions of $P(x) = 0$.

iii



The graph of $P(x) = 2x^3 - 3x^2 - 11 + 6$.